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Implications of $\mathcal{N}=1$ Superconformal Symmetry for Chiral Fields

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The requirements of $\mathcal{N}=1$ superconformal invariance for the correlation functions of chiral superfields are analysed. Complete expressions are found for the three point function for the general spin case and for the four point function for scalar superfields for $\sum q_i = 3$ where q_i is the scale dimension for the i'th superfield and is related to the $U(1)$ R-charge. In the latter case the relevant Ward identities reduce to eight differential equations for four functions of u, v which are invariants when the superconformal symmetry is reduced to the usual conformal group. The differential equations have a general solution given by four linearly independent expressions involving a two variable generalisation of the hypergeometric function. By considering the behaviour under permutations, or crossing symmetry, the chiral four point function is shown to be determined up to a single overall constant. The results are in accord with the supersymmetric operator product expansion.

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1. Introduction

It is now clear that there exist in four, and also five and six, dimensions a plethora of non trivial quantum field theories, without any coupling to gravity, which possess fixed points realising conformal symmetry. The evidence for such conformally invariant theories is strongest in the case where the field theory is supersymmetric and the conformal group is extended to the superconformal group. In four dimensions this may be identified as $SU(2,2|\mathcal{N})$ where the cases of $\mathcal{N}=1,2,4$ are relevant for renormalisable field theories, although for $\mathcal{N}=4$ the group contains an ideal so the group may be reduced to the projective group $PSU(2,2|4)$. Besides the usual 15 parameter conformal group the superconformal group also contains an R-symmetry, $U(1)$, $U(2)$, and $SU(4)$ for $\mathcal{N}=1,2$ and 4. The case of $\mathcal{N}=4$ gauge theories has been known for a long time since these have vanishing β-functions and recently the strong coupling limit of such theories has been explored through the ADS/CFT correspondence. Nevertheless $\mathcal{N}=1$ theories may possess, for appropriate matter content, a conformal window with superconformal invariant fixed points.

Such theories are naturally described in terms of superfields defined over a superspace involving Grassman variables. In general the action of conformal transformations on the coordinates is non linear and the construction of covariant correlation functions is not as straightforward as realising the consequencies of symmetry under the usual Poincaré group transformations which act linearly on the coordinates. A formalism which allows the construction of two and three point, and in principle $n>3$ as well, correlation functions was described in [1] for the $\mathcal{N}=1$ superconformal group in four dimensions, extending a similar discussion for the ordinary conformal group in [2,3]. This has been extended to six dimensions and also to $\mathcal{N}>1$ in four dimensions by [4] and this approach has been used for an analysis of particular two and three point functions for $\mathcal{N}=2$ in [5]. In four dimensions the superconformal group is compatible with acting on superfields restricted to the chiral projections, $z_+=(x_+,\theta)$ where θ^α is a chiral spinor, or $z_-=(x_-,\bar\theta)$, where $\bar\theta^{\dot\alpha}$ is an anti-chiral spinor. For two points $z_1=(x_1,\theta_1,\bar\theta_1)$ and $z_2=(x_2,\theta_2,\bar\theta_2)$ then the constructions described in [1] are expressed in terms of $\tilde{\mathrm{x}}_{\bar 12}=x_{\bar 12}{}^a\tilde\sigma_a$ which is constructed from z_{1-} and z_{2+} and transforms homogeneously under superconformal transformations with a local scale transformation and rotation at z_1 and also at z_2, in particular

$$x_{\bar 12}{}^2 \longrightarrow \frac{x_{\bar 12}{}^2}{\bar\Omega(z_{1-})\Omega(z_{2+})}\,, \tag{1.1}$$

which is a direct generalisation of the transformation of $(x_1-x_2)^2$ under the usual conformal group. For three points in superspace z_1,z_2,z_3 it is also possible to generalise the treatment for forming conformally covariant expressions [2,3] by introducing the variables X_1,Θ_1

and their conjugates $\bar{X}_1, \bar{\Theta}_1$, which are related in the same fashion as x_+, θ and $x_-, \bar{\theta}$, that transform homogeneously under local rotations and scale transformations at z_1. In particular ${X_1}^2 = {x_{\bar{2}3}}^2/({x_{\bar{2}1}}^2\, {x_{\bar{1}3}}^2)$ and its transformation follows from (1.1). These variables play an essential role in the general construction of three point functions described in [1].

However X_1 depends on z_1, z_{2-}, z_{3+} while Θ_1 is formed from z_1, z_{2-}, z_{3-} and correspondingly for their conjugates. In consequence the construction of correlation functions for chiral fields alone, which depend only on the z_+'s, requires cancellations of unwanted chiral components in the formalism of [1]. This was achieved quite easily for the three point function for chiral scalar superfields $\langle \phi_1(z_{1+})\phi_2(z_{2+})\phi_3(z_{3+})\rangle$ but efforts to generalise such a treatment to the four point function became very involved and were not carried through to a conclusion. In a different approach Pickering and West [6] obtained identities expressing superconformal invariance directly for three and four point functions which from the start depend solely on the z_+'s.[1] The three point case was again rather simple but the invariance for the four point function led to a system of eight linear first order partial differential equations for four functions of two variables u, v which are the two independent cross ratios formed from $(x_{i+} - x_{j+})^2$. Without supersymmetry u, v are conformal invariants and the general conformally covariant four point function involves an arbitrary function of u, v.

In this paper we follow a similar path to finding the conditions following from $\mathcal{N} = 1$ superconformal invariance for chiral superfield correlation functions. For such superfields the scale dimension q is trivially related to the R-charge $3q$ and they may belong only to $(j, 0)$ spinor representations of the four dimensional Lorentz group. For a subgroup of $Sl(4|1)$, the complexification of $SU(2,2|1)$, which contains the conformal group, the constraints are simply realised where by introducing, for three points z_{1+}, z_{2+}, z_{3+}, spinor variables Λ which transform homogeneously. Extending this to the full superconformal group involves further constraints. For the three point function we must require $\sum q_i = 3$ and for the four point function, with the same condition on the q_i's, we obtain eight differential equations similar to those found by Pickering and West. These are here solved in terms of a certain two variable generalisation of the hypergeometric function F_4. The superconformal symmetry identities for the correlation function for four scalar chiral superfields have in general four linearly independent solutions. A careful analysis of the behaviour under permutations of the superfields or imposing crossing symmetry, which requires taking into account non trivial identities for the F_4 functions under analytic continuation, leads to a unique expression up to an overall constant.

In this paper in the next section we describe how the the superconformal group acts on the chiral coordinates z_+ and obtain the essential transformation formulae used to

[1] The three point function was considered earlier by Conlong and West [7].

construct superconformal correlation functions later. In section three we derive the superconformal identities for the three and four point functions for chiral fields. Results for the three point function are obtained for arbirary spin. For the four point function we restrict to scalar fields and the eight equations for four functions of u, v are obtained. These equations have four linearly independent solutions which are expressed in terms of the functions F_4 mentioned above. The conditions necessary for Bose symmetry or crossing symmetry are analysed in section 4. By requiring independence under the path of analytic continuation, and using the transformation formulae for the F_4 function, a unique result, up to a single overall constant, is obtained. In section 5 an alternative expression in terms of a function G, defined in terms of two F_4 functions, is obtained. This allows the short distance limits of the four point function to become manifest. In section 6 we endeavour to count the constraints for higher point functions of chiral scalar superfields. At least for six and higher point functions the differential equations arising from superconformal invariance do not fully determine the functions arising in the general expansion. In section 7 we consider the application of the operator product expansion for two chiral superfields to the four point function. The contribution of a chiral scalar superfield in the operator product expansion to the four point function is obtained in a form which satisfies all the superconformal Ward identities but which does not satisfy the crossing symmetry relations by itself. In section 8 we show how an integral over superspace defines a superconformal N-point function for any N and this is shown to be in accord with the results obtained in section 3 for $N = 3, 4$. In appendix A we discuss an alternative derivation of the essential equations for the four point function based on using superconformal transformations to fix two points. In appendix B we describe some results relevant in section 8 for the evaluation of conformal integrals while in appendix C some results which allow for the simplification of the two variable function G used here are described. For the cases of relevance here it is shown how they may be reduced to sums of products of ordinary hypergeometric functions. In appendix D we recapitulate some results for the operator product expansion which are applied to the chiral superfield four point function in appendix E.

2. Superconformal Transformations on Chiral Superspace

We use the standard identification of 4-vectors with 2×2 matrices[2] so that the action

[2] The notation is identical with [1] and is essentially that of of Wess and Bagger [8]. Thus θ^α, $\bar\theta^{\dot\alpha}$ are regarded as row, column vectors and we let $\tilde\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta$, $\tilde{\bar\theta}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}$ form associated column, row vectors, $\theta^2 = \theta\tilde\theta$, $\bar\theta^2 = \tilde{\bar\theta}\bar\theta$. The basis of 2×2-matrices is given by the hermitian σ-matrices σ_a, $\tilde\sigma_a$, $\sigma_{(a}\tilde\sigma_{b)} = -\eta_{ab}1$, and for a 4-vector x^a then $\mathrm{x}_{\alpha\dot\alpha} = x^a(\sigma_a)_{\alpha\dot\alpha}$, $\tilde{\mathrm{x}}^{\dot\alpha\alpha} = x^a(\tilde\sigma_a)^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\mathrm{x}_{\beta\dot\beta}$, with inverse $x^a = -\frac{1}{2}\mathrm{tr}(\sigma^a\tilde{\mathrm{x}})$.

of infinitesimal superconformal transformations on z_+ is given by [9,1]

$$\begin{aligned}
\delta\tilde{\mathrm{x}}_+ &= \tilde{a} + \bar{\omega}\tilde{\mathrm{x}}_+ - \tilde{\mathrm{x}}_+\omega + (\kappa+\bar{\kappa})\tilde{\mathrm{x}}_+ + \tilde{\mathrm{x}}_+\mathrm{b}\tilde{\mathrm{x}}_+ + 4i\,\bar{\epsilon}\theta - 4\,\tilde{\mathrm{x}}_+\eta\theta\,, \\
\delta\theta &= \epsilon - \theta\omega + \kappa\,\theta + \theta\mathrm{b}\tilde{\mathrm{x}}_+ - i\bar{\eta}\,\tilde{\mathrm{x}}_+ + 2\,\tilde{\eta}\,\theta^2\,, \\
\omega_\beta{}^\alpha &= \, -\tfrac{1}{4}\omega^{ab}(\sigma_a\tilde{\sigma}_b)_\beta{}^\alpha\,, \qquad \bar{\omega}^{\dot\alpha}{}_{\dot\beta} = -\tfrac{1}{4}\omega^{ab}(\tilde{\sigma}_a\sigma_b)^{\dot\alpha}{}_{\dot\beta}\,,
\end{aligned} \tag{2.1}$$

where a^a corresponds to a translation, $\omega^{ab} = -\omega^{ba}$ an infinitesimal rotation, b^a a special conformal transformation, $\kappa+\bar{\kappa}$ a rescaling and $(\kappa-\bar{\kappa})/i$ a $U(1)_R$ phase while $\epsilon^\alpha, \bar{\epsilon}^{\dot\alpha}$ are supertranslations with $\eta_\alpha, \bar{\eta}_{\dot\alpha}$ their superconformal extensions. These parameters may be written as elements of a supermatrix

$$M = \begin{pmatrix} \omega - \frac{1}{3}(\kappa+2\bar{\kappa})1 & -i\mathrm{b} & 2\eta \\ -i\tilde{\mathrm{a}} & \bar{\omega} + \frac{1}{3}(2\kappa+\bar{\kappa})1 & 2\bar{\epsilon} \\ 2\epsilon & 2\bar{\eta} & \frac{2}{3}(\kappa-\bar{\kappa}) \end{pmatrix}\,, \tag{2.2}$$

so that

$$\delta_2\delta_1 - \delta_1\delta_2 = \delta_3 \qquad \Rightarrow \qquad [M_1, M_2] = M_3\,. \tag{2.3}$$

Since $\mathrm{str}\, M = 0$, M is a generator of $Sl(4|1)$ which is reduced to $SU(2,2|1)$ by imposing the appropriate reality condition.

For two points z_{i+}, z_{j+} we may define

$$\tilde{\mathrm{x}}_{ij} = \tilde{\mathrm{x}}_{i+} - \tilde{\mathrm{x}}_{j+}\,, \qquad \ell_{ij} = (\theta_i - \theta_j)\tilde{\mathrm{x}}_{ij}{}^{-1}\,, \quad \tilde{\mathrm{x}}_{ij}{}^{-1} = \mathrm{x}_{ji}/x_{ij}{}^2\,, \tag{2.4}$$

which, from (2.1), transform as

$$\begin{aligned}
\delta\tilde{\mathrm{x}}_{ij} &= \big(\bar{\omega} + \bar{\kappa} + \tilde{\mathrm{x}}_{i+}\mathrm{b} + 4i\,\bar{\epsilon}_i\ell_{ij}\big)\tilde{\mathrm{x}}_{ij} + \tilde{\mathrm{x}}_{ij}\big(-\omega + \kappa + \mathrm{b}\tilde{\mathrm{x}}_{j+} - 4\,\eta\theta_j\big)\,, \\
\delta\ell_{ij} &= \, -\ell_{ij}\big(\bar{\omega} + \bar{\kappa} + \tilde{\mathrm{x}}_{i+}\mathrm{b} - 4\,\theta_i\eta\big) - i\bar{\eta} + \theta_i\mathrm{b} + 2i\,\ell_{ij}{}^2\,\tilde{\bar{\epsilon}}_i\,,
\end{aligned} \tag{2.5}$$

where we also define

$$\bar{\epsilon}(x_+) = \bar{\epsilon} + i\,\mathrm{x}_+\bar{\eta}\,, \qquad \bar{\epsilon}_i = \bar{\epsilon}(x_{i+})\,. \tag{2.6}$$

Furthermore if

$$r_{ij} = (x_{i+} - x_{j+})^2 = -\det(\tilde{\mathrm{x}}_{ij})\,, \tag{2.7}$$

defining also the spinor $\Lambda_{i(jk)}$ and 4-vector $Y_{i(jk)}$ by

$$\Lambda_{i(jk)} = \ell_{ij} - \ell_{ik}\,, \qquad \mathrm{Y}_{i(jk)} = -\tilde{\mathrm{x}}_{ij}{}^{-1} + \tilde{\mathrm{x}}_{ik}{}^{-1} = \tilde{\mathrm{x}}_{ji}{}^{-1}\tilde{\mathrm{x}}_{jk}\,\tilde{\mathrm{x}}_{ik}{}^{-1} = -\tilde{\mathrm{x}}_{ki}{}^{-1}\tilde{\mathrm{x}}_{kj}\,\tilde{\mathrm{x}}_{ij}{}^{-1}\,, \tag{2.8}$$

we then have

$$\delta r_{ij} = 2\big(\kappa + \bar{\kappa} - b{\cdot}x_i - b{\cdot}x_j + 2\theta_j\eta - 2i\,\ell_{ij}\bar{\epsilon}_i\big)r_{ij}\,, \tag{2.9}$$

and

$$
\begin{aligned}
\delta\Lambda_{i(jk)} = {} & -\Lambda_{i(jk)}\big(\bar\omega + \bar\kappa + \tilde{\rm x}_{i+}{\rm b} + 4\,\theta_i\eta - 2i\,(\tilde\ell_{ij} + \tilde\ell_{ik})\tilde{\bar\epsilon}_i\big)\,, \\
\delta{\rm Y}_{i(jk)} = {} & \big(\omega - \kappa - {\rm b}\tilde{\rm x}_{i+} + 4\eta\theta_i\big){\rm Y}_{i(jk)} \\
& \quad - {\rm Y}_{i(jk)}\big(\bar\omega + \bar\kappa + \tilde{\rm x}_{i+}{\rm b} + 4i\,\bar\epsilon_i\ell_{ij} - 4i\,\tilde{\rm x}_{ij}\tilde{\rm x}_{kj}{}^{-1}\bar\epsilon_k\,\Lambda_{i(jk)}\big)\,.
\end{aligned}
\tag{2.10}
$$

If we set $\bar\epsilon, \eta = 0$ then it is evident from (2.5) that $\tilde{\rm x}_{ij}$ varies homogeneously with local scale transformations and rotations at z_{i+} and z_{j+}. Similarly the variations of $\Lambda_{i(jk)}$ and $Y_{i(jk)}$ given by (2.10) then corresponds to a local scale transformation and rotation at z_{i+}. The group $G_0 \subset Sl(4|1)$ generated by matrices M as in (2.2) with $\bar\epsilon, \eta = 0$ contains the usual bosonic conformal group and it is straightforward to construct covariantly transforming correlation functions and also invariants under G_0. Thus for four points $z_{1+}, z_{2+}, z_{3+}, z_{4+}$ we may define the G_0 invariant cross ratios

$$
u = \frac{r_{12}\, r_{34}}{r_{13}\, r_{24}}\,, \qquad\qquad v = \frac{r_{14}\, r_{23}}{r_{13}\, r_{24}}\,. \tag{2.11}
$$

For later use we also define unit 4-vectors by

$$
\hat{\rm x}_{ji} = {\rm x}_{ji}\, r_{ij}{}^{-\frac12} = \tilde{\rm x}_{ij}{}^{-1} r_{ij}{}^{\frac12}\,, \qquad \hat{\rm Y}_{i(jk)} = {\rm Y}_{i(jk)}\bigg(\frac{r_{jk}}{r_{ij}\, r_{ik}}\bigg)^{-\frac12}\,, \tag{2.12}
$$

so that, from (2.5) and (2.10), we have

$$
\begin{aligned}
\delta\hat{\rm x}_{ji} = {} & \hat\omega_j\hat{\rm x}_{ji} - \hat{\rm x}_{ji}\big(\bar\omega + \tfrac12(\tilde{\rm x}_i{\rm b} - \tilde{\rm b}{\rm x}_i) + 4i(\bar\epsilon_i\ell_{ij} + \tfrac12\ell_{ij}\bar\epsilon_i\,1)\big)\,, \\
\delta\hat{\rm Y}_{i(jk)} = {} & \hat\omega_i\hat{\rm Y}_{i(jk)} - \hat{\rm Y}_{i(jk)}\big(\bar\omega + \tfrac12(\tilde{\rm x}_i{\rm b} - \tilde{\rm b}{\rm x}_i) + 4i(\bar\epsilon_i\ell_{ij} + \tfrac12\ell_{ij}\bar\epsilon_i\,1) \\
& \qquad - 4i\,(\tilde{\rm x}_{ij}\tilde{\rm x}_{kj}{}^{-1}\bar\epsilon_k\Lambda_{i(jk)} + \tfrac12\Lambda_{i(jk)}\tilde{\rm x}_{ij}\tilde{\rm x}_{kj}{}^{-1}\bar\epsilon_k\,1)\big)\,,
\end{aligned}
\tag{2.13}
$$

where $\hat\omega_\alpha{}^\beta(z_+)$, $\hat\omega_\alpha{}^\alpha = 0$, is given by

$$
\hat\omega(z_+) = \omega - \tfrac12({\rm b}\tilde{\rm x}_+ - {\rm x}_+\tilde{\rm b}) + 4(\eta\theta + \tfrac12\theta\eta\,1)\,, \qquad \hat\omega_i = \hat\omega(z_{i+})\,. \tag{2.14}
$$

In obtaining (2.13) it is useful to note, with the definitions in (2.6),

$$
\bar\epsilon_i = \tilde{\rm x}_{ij}\tilde{\rm x}_{kj}{}^{-1}\bar\epsilon_k + \tilde{\rm x}_{ik}\tilde{\rm x}_{jk}{}^{-1}\bar\epsilon_j\,. \tag{2.15}
$$

To achieve symmetry under the full superconformal group requires cancellation of all terms involving $\bar\epsilon_i$. This requires further constraints. To obtain the necessary conditions we first obtain from (2.10),

$$
\delta\Lambda_{i(jk)}{}^2 = -2\big(\bar\kappa - b{\cdot}x_{i+} + 4\,\theta_i\eta + i(\ell_{ij} + \ell_{ik})\bar\epsilon_i\big)\Lambda_{i(jk)}{}^2\,, \tag{2.16}
$$

so that, using (2.9), we have

$$\delta\,\frac{{\Lambda_{i(jk)}}^2}{r_{ij}} = -2(2\sigma_i+\sigma_j)\frac{{\Lambda_{i(jk)}}^2}{r_{ij}}\,, \tag{2.17}$$

with

$$\sigma(z_+) = \tfrac{1}{3}(\kappa+2\bar\kappa) - b{\cdot}x_+ + 2\,\theta\eta\,, \qquad \sigma_i = \sigma(z_{i+})\,. \tag{2.18}$$

In (2.17) the terms involving $\bar\epsilon_i$ have therefore cancelled. For subsequent use we also note that, from (2.9) and (2.18),

$$\delta\,\frac{r_{ij}}{r_{ik}} = 2\big(\sigma_j - \sigma_k - 2i\,\Lambda_{i(jk)}\bar\epsilon_i\big)\frac{r_{ij}}{r_{ik}}\,. \tag{2.19}$$

From the definition in (2.7) we have

$$\Lambda_{i(jk)} = \Lambda_{i(jl)} + \Lambda_{i(lk)}\,, \qquad \Lambda_{i(jk)} = -\Lambda_{i(kj)}\,, \tag{2.20}$$

as well as

$$\Lambda_{i(jk)} = \Lambda_{j(ik)}\tilde{\mathrm{x}}_{jk}{\tilde{\mathrm{x}}_{ik}}{}^{-1}\,. \tag{2.21}$$

This leads to

$$\frac{{\Lambda_{i(jk)}}^2}{r_{jk}} = \frac{{\Lambda_{j(ki)}}^2}{r_{ik}} = \frac{{\Lambda_{k(ij)}}^2}{r_{ij}}\,, \tag{2.22}$$

so that ${\Lambda_{i(jk)}}^2/r_{jk}$ is completely symmetric. This is manifest in the transformation properties, obtained from (2.17) and (2.19),

$$\delta\,\frac{{\Lambda_{i(jk)}}^2}{r_{jk}} = -2(\sigma_i+\sigma_j+\sigma_k)\frac{{\Lambda_{i(jk)}}^2}{r_{jk}}\,. \tag{2.23}$$

These results allow the construction of superconformal covariant expressions for chiral superfield correlation functions. For a quasi-primary spin j chiral field belonging to the $(j,0)$ representation $\phi_I = \phi_{\alpha_1\dots\alpha_{2j}}$, totally symmetric in $\alpha_1\dots\alpha_{2j}$ so that I takes $2j+1$ values, we have, with $\hat\omega$ and σ defined in (2.14) and (2.18),

$$\begin{gathered}\delta\phi_I(z_+) = \,-\mathcal{L}_{z_+}\phi_I(z_+) - 2q\,\sigma(z_+)\phi_I(z_+) + (\hat\omega(z_+){\cdot}s)_I{}^J\phi_J(z_+)\,,\\ \mathcal{L}_{z_+} = \delta z_+{\cdot}\frac{\partial}{\partial z_+}\,, \qquad (\hat\omega{\cdot}s)_I{}^J\phi_J = 2j\,\hat\omega_{(\alpha_1}{}^\beta\phi_{\alpha_2\dots\alpha_{2j})\beta}\,,\end{gathered} \tag{2.24}$$

where $\hat\omega{\cdot}s = \hat\omega_\beta{}^\alpha s_\alpha{}^\beta$ with $s_\alpha{}^\beta$, $[s_\alpha{}^\beta, s_\gamma{}^\delta] = \delta_\alpha{}^\delta s_\gamma{}^\beta - \delta_\gamma{}^\beta s_\alpha{}^\delta$, generators for spin j. The representations are specified by (j,q) and for these to be unitary with positive energy either $j=q=0$, which is the trivial singlet case, or $q \geq j+1$, with q determining both the R-charge and scale dimension.

3. Chiral Three and Four Point Functions

The superconformal Ward identities for the n-point chiral correlation function of chiral superfields belonging to representations (j_i, q_i), $i = 1, \dots n$, requires

$$\begin{aligned}\sum_i \big(\mathcal{L}_i + 2q_i\,\sigma_i\big)\big\langle\phi_{I_1}(z_{1+})\dots\phi_{I_n}(z_{n+})\big\rangle \qquad\qquad\\ - \sum_i (\hat\omega_i{\cdot}s_i)_{I_i}{}^J\big\langle\phi_{I_1}(z_{1+})\dots\phi_J(z_{i+})\dots\phi_{I_n}(z_{n+})\big\rangle = 0\,,\end{aligned} \tag{3.1}$$

with s_i the generators for spin j_i. We solve these identities explicitly for general spins when $n = 3$ and for scalar chiral superfields when $n = 4$ so long as

$$\sum_i q_i = 3\,. \tag{3.2}$$

For the three point function we may write

$$\big\langle\phi_{I_1}(z_{1+})\phi_{I_2}(z_{2+})\phi_{I_3}(z_{3+})\big\rangle = \frac{{\Lambda_{1(23)}}^2}{r_{23}}\left(\frac{r_{12}}{r_{13}}\right)^{1-q_2}\left(\frac{r_{12}}{r_{23}}\right)^{1-q_1} F_{I_1I_2I_3}(x_{1+}, x_{2+}, x_{3+})\,. \tag{3.3}$$

For scalar fields $F_{I_1I_2I_3} \to C_{123}$, a constant, and it is easy to see from (2.23) and (2.19), since ${\Lambda_{1(23)}}^3 = 0$, that this has the correct transformation properties in accord with (3.1) and also from (2.22), with (3.2), it is completely symmetric under simultaneous permutations of z_{1+}, z_{2+}, z_{3+} and q_1, q_2, q_3. To construct $F_{I_1I_2I_3}$ for non zero spins we introduce

$$\mathcal{I}^{(j)}(\hat x)_{I\bar I} = \hat{\mathrm{x}}_{(\alpha_1|\dot\alpha_1}\dots\hat{\mathrm{x}}_{\alpha_{2j})\dot\alpha_{2j}}\,, \tag{3.4}$$

which is a bi-spinor, transforming as $(j,0)\times(0,j)$. The appropriate solution is then

$$F_{I_1I_2I_3}(x_{1+}, x_{2+}, x_{3+}) = \mathcal{I}^{(j_1)}(\hat Y_{1(23)})_{I_1\bar I_1}\mathcal{I}^{(j_2)}(\hat x_{21})_{I_2\bar I_2}\mathcal{I}^{(j_3)}(\hat x_{31})_{I_3\bar I_3}\, t^{\bar I_1\bar I_2\bar I_3}_{j_1j_2j_3}\,, \tag{3.5}$$

where $\hat x_{21}, \hat x_{31}$ and $\hat Y_{1(23)}$ are given by (2.13) and $t^{\bar I_1\bar I_2\bar I_3}_{j_1j_2j_3}$ is an invariant tensor satisfying, for any rotation r,

$$D^{(j_1)}(r)^{\bar I_1}{}_{\bar J_1}D^{(j_2)}(r)^{\bar I_2}{}_{\bar J_2}D^{(j_3)}(r)^{\bar I_3}{}_{\bar J_3}\,t^{\bar J_1\bar J_2\bar J_3}_{j_1j_2j_3} = t^{\bar I_1\bar I_2\bar I_3}_{j_1j_2j_3}\,. \tag{3.6}$$

Such tensors are given by the usual Clebsch-Gordan coefficients if $|j_2 - j_3| \le j_1 \le j_2 + j_3$. Infinitesimally (3.6) requires $(\bar\omega{\cdot}\bar s_1)^{\bar I_1}{}_{\bar J}\,t^{\bar J\bar I_2\bar I_3}_{j_1j_2j_3} + (\bar\omega{\cdot}\bar s_2)^{\bar I_2}{}_{\bar J}\,t^{\bar I_1\bar J\bar I_3}_{j_1j_2j_3} + (\bar\omega{\cdot}\bar s_3)^{\bar I_3}{}_{\bar J}\,t^{\bar I_1\bar I_2\bar J}_{j_1j_2j_3} = 0$, for any $\bar\omega^{\dot\alpha}{}_{\dot\beta}$, $\bar\omega^{\dot\alpha}{}_{\dot\alpha} = 0$, and using this with (2.13), and since terms $\propto \Lambda_{1(23)}$ in (2.13)

vanish, it is straightforward to see that this ensures that (3.5) has the correct spinorial transformation properties according to (3.1). Although the construction in (3.5) is apparently asymmetric we may re-express the result in the equivalent form expected by permutation symmetry. Using (3.6) first with $D^{(j)}(r) = \mathcal{I}^{(j)}(\hat{x}_{32})^{-1}\mathcal{I}^{(j)}(\hat{x}_{31})$ then we have $\mathcal{I}^{(j_1)}(\hat{Y}_{1(23)}) = \mathcal{I}^{(j_1)}(\hat{x}_{12})D^{(j_1)}(r)$, since $\hat{\mathrm{Y}}_{1(23)} = \hat{\mathrm{x}}_{12}\hat{\mathrm{x}}_{32}{}^{-1}\hat{\mathrm{x}}_{31}$, and similarly $\mathcal{I}^{(j_2)}(\hat{Y}_{2(13)}) = \mathcal{I}^{(j_2)}(\hat{x}_{21})D^{(j_2)}(r)^{-1}$, or secondly with $D^{(j)}(r) = \mathcal{I}^{(j)}(\hat{x}_{23})^{-1}\mathcal{I}^{(j)}(\hat{x}_{21})$ when we have $\mathcal{I}^{(j_1)}(\hat{Y}_{1(23)}) = (-1)^{2j_1}\mathcal{I}^{(j_1)}(\hat{x}_{13})D^{(j_1)}(r)$, allows the expression (3.5) also to be written as

$$\begin{aligned} F_{I_1I_2I_3}(x_{1+},x_{2+},x_{3+}) &= \mathcal{I}^{(j_1)}(\hat{x}_{12})_{I_1\bar{I}_1}\mathcal{I}^{(j_2)}(\hat{Y}_{2(13)})_{I_2\bar{I}_2}\mathcal{I}^{(j_3)}(\hat{x}_{32})_{I_3\bar{I}_3}\, t^{\bar{I}_1\bar{I}_2\bar{I}_3}_{j_1j_2j_3}\,, \\ &= \mathcal{I}^{(j_1)}(\hat{x}_{13})_{I_1\bar{I}_1}\mathcal{I}^{(j_2)}(\hat{x}_{23})_{I_2\bar{I}_2}\mathcal{I}^{(j_3)}(\hat{Y}_{3(12)})_{I_3\bar{I}_3}\,(-1)^{2j_1}t^{\bar{I}_1\bar{I}_2\bar{I}_3}_{j_1j_2j_3}\,. \end{aligned} \tag{3.7}$$

Fermi/Bose symmetry following from $\phi_1(z_{1+})\phi_2(z_{2+}) = P_{12}\phi_2(z_{2+})\phi_1(z_{1+})$, where $P_{12} = -1$ for both $2j_1, 2j_2$ odd, and with similar definitions of P_{23}, P_{13}, requires then

$$P_{23}t^{\bar{I}_1\bar{I}_3\bar{I}_2}_{j_1j_3j_2} = (-1)^{2j_1}t^{\bar{I}_1\bar{I}_2\bar{I}_3}_{j_1j_2j_3}\,, \quad P_{12}t^{\bar{I}_2\bar{I}_1\bar{I}_3}_{j_2j_1j_3} = t^{\bar{I}_1\bar{I}_2\bar{I}_3}_{j_1j_2j_3}\,, \quad P_{13}P_{23}t^{\bar{I}_3\bar{I}_1\bar{I}_2}_{j_3j_1j_2} = (-1)^{2j_1}t^{\bar{I}_1\bar{I}_2\bar{I}_3}_{j_1j_2j_3}\,. \tag{3.8}$$

If $(j_1,q_1) = (j_2,q_2)$ this provides constraints on possible values of j_3, and similarly for any other identical pair.

The discussion of the four point function is more involved and so is restricted here to chiral scalar superfields ϕ_i, $i = 1,2,3,4$. As shown in the section 6, and in accord with the analysis of Pickering and West [6], we may restrict to an expansion in a nilpotent basis formed by ${\Lambda_{i(jk)}}^2$, $1 \le i < j < k \le 4$, with coefficients involving functions of the cross ratios u, v defined in (2.11). Thus we write,

$$\begin{aligned} &\big\langle \phi_1(z_{1+})\phi_2(z_{2+})\phi_3(z_{3+})\phi_4(z_{4+})\big\rangle \\ &\quad = \frac{{\Lambda_{1(23)}}^2}{r_{23}}\bigg(\frac{r_{12}}{r_{13}}\bigg)^{q_3-1}\bigg(\frac{r_{23}}{r_{13}}\bigg)^{q_1+q_4-1}\bigg(\frac{r_{12}}{r_{24}}\bigg)^{q_4} f_{123}(u,v) \\ &\qquad + \frac{{\Lambda_{1(24)}}^2}{r_{24}}\bigg(\frac{r_{14}}{r_{24}}\bigg)^{q_2-1}\bigg(\frac{r_{12}}{r_{24}}\bigg)^{q_3+q_4-1}\bigg(\frac{r_{14}}{r_{13}}\bigg)^{q_3} f_{124}(u,v) \\ &\qquad + \frac{{\Lambda_{1(34)}}^2}{r_{34}}\bigg(\frac{r_{34}}{r_{13}}\bigg)^{q_1-1}\bigg(\frac{r_{14}}{r_{13}}\bigg)^{q_2+q_3-1}\bigg(\frac{r_{34}}{r_{24}}\bigg)^{q_2} f_{134}(u,v) \\ &\qquad + \frac{{\Lambda_{2(34)}}^2}{r_{34}}\bigg(\frac{r_{23}}{r_{24}}\bigg)^{q_4-1}\bigg(\frac{r_{34}}{r_{24}}\bigg)^{q_1+q_2-1}\bigg(\frac{r_{23}}{r_{13}}\bigg)^{q_1} f_{234}(u,v)\,. \end{aligned} \tag{3.9}$$

It is easy to see that this has the correct form to satisfy the superconformal Ward identities up to terms proportional to $\bar{\epsilon}_i$ which are generated by the variation of u, v and also by the variation of the last factor of r_{ij}/r_{ik} in each of the four terms appearing in the expression

for the four point function given by (3.9). Such terms must cancel. To achieve a linearly independent basis we restrict all contributions to only $\bar\epsilon_1, \bar\epsilon_2$, by using (2.15), and also to involve just ${\Lambda_{1(23)}}^2\Lambda_{1(24)}$ and ${\Lambda_{1(24)}}^2\Lambda_{1(23)}$. Using (2.19) and (2.21) we may find

$$\begin{aligned}\delta v = {}& -4iv(\Lambda_{1(43)}\bar\epsilon_1 + \Lambda_{2(34)}\bar\epsilon_2)\\ = {}& -4iv\big(\Lambda_{1(23)}(\bar\epsilon_1 - \tilde{\mathrm{x}}_{13}{\tilde{\mathrm{x}}_{23}}^{-1}\bar\epsilon_2) - \Lambda_{1(24)}(\bar\epsilon_1 - \tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\bar\epsilon_2)\big)\,,\end{aligned} \tag{3.10}$$

and also

$$\begin{aligned}\delta u = {}& -4iu(\Lambda_{1(23)}\bar\epsilon_1 + \Lambda_{4(32)}\bar\epsilon_4)\\ = {}& -4i\big(\Lambda_{1(23)} - \Lambda_{1(24)}\big)\big(u\bar\epsilon_1 - v\,\tilde{\mathrm{x}}_{13}{\tilde{\mathrm{x}}_{23}}^{-1}\bar\epsilon_2 + \tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\bar\epsilon_2 - u\,\tilde{\mathrm{x}}_{13}{\tilde{\mathrm{x}}_{43}}^{-1}\tilde{\mathrm{x}}_{42}{\tilde{\mathrm{x}}_{12}}^{-1}\bar\epsilon_1\big)\\ & -4iu\,\Lambda_{1(24)}\tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\bar\epsilon_2\,.\end{aligned} \tag{3.11}$$

It is easy to see that

$${\Lambda_{1(34)}}^2\Lambda_{1(23)} = {\Lambda_{1(34)}}^2\Lambda_{1(24)} = {\Lambda_{1(23)}}^2\Lambda_{1(24)} + {\Lambda_{1(24)}}^2\Lambda_{1(23)}\,, \tag{3.12}$$

and, from (3.10) and (3.11), we have

$$\begin{aligned}{\Lambda_{2(34)}}^2\delta v = {}& 4iv\,\frac{r_{13}}{r_{23}}{\Lambda_{1(23)}}^2\Lambda_{1(24)}\big(\bar\epsilon_1 - \tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\tilde{\mathrm{x}}_{23}{\tilde{\mathrm{x}}_{13}}^{-1}\bar\epsilon_1\big)\\ & -4iv\,\frac{r_{14}}{r_{24}}{\Lambda_{1(24)}}^2\Lambda_{1(23)}\big(\bar\epsilon_1 - \tilde{\mathrm{x}}_{13}{\tilde{\mathrm{x}}_{23}}^{-1}\tilde{\mathrm{x}}_{24}{\tilde{\mathrm{x}}_{14}}^{-1}\bar\epsilon_1\big)\,,\end{aligned} \tag{3.13}$$

and

$${\Lambda_{2(34)}}^2\delta u = -4iu\,\frac{r_{13}}{r_{23}}{\Lambda_{1(23)}}^2\Lambda_{1(24)}\tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\tilde{\mathrm{x}}_{23}{\tilde{\mathrm{x}}_{13}}^{-1}\bar\epsilon_1 - 4iu\,\frac{r_{14}}{r_{24}}{\Lambda_{1(24)}}^2\Lambda_{1(23)}\bar\epsilon_1\,. \tag{3.14}$$

To complete the calculation we need the contributions from the variations of the relevant r_{ij}/r_{ik} factors in the four terms appearing in (3.9). These are given respectively by

$$\begin{aligned}-4iq_4{\Lambda_{1(23)}}^2\Lambda_{2(14)}\bar\epsilon_2 = {}& -4iq_4{\Lambda_{1(23)}}^2\Lambda_{1(24)}\tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\bar\epsilon_2\,,\\ -4iq_3{\Lambda_{1(24)}}^2\Lambda_{1(43)}\bar\epsilon_1 = {}& -4iq_3{\Lambda_{1(24)}}^2\Lambda_{1(23)}\bar\epsilon_1\,,\\ -4iq_2{\Lambda_{1(34)}}^2\Lambda_{4(32)}\bar\epsilon_4 = {}& 4iq_2\big({\Lambda_{1(23)}}^2\Lambda_{1(24)} + {\Lambda_{1(24)}}^2\Lambda_{1(23)}\big)\big(\bar\epsilon_1 - \tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\bar\epsilon_2\big)\,,\\ -4iq_1{\Lambda_{2(34)}}^2\Lambda_{3(21)}\bar\epsilon_3 = {}& -4iq_1\,\frac{r_{13}}{r_{23}}{\Lambda_{1(23)}}^2\Lambda_{1(24)}\big(\tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\tilde{\mathrm{x}}_{23}{\tilde{\mathrm{x}}_{13}}^{-1}\bar\epsilon_1 - \tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\bar\epsilon_2\big)\\ & -4iq_1\,\frac{r_{14}}{r_{24}}{\Lambda_{1(24)}}^2\Lambda_{1(23)}\big(\bar\epsilon_1 - \tilde{\mathrm{x}}_{13}{\tilde{\mathrm{x}}_{23}}^{-1}\bar\epsilon_2\big)\,.\end{aligned} \tag{3.15}$$

Using in addition the identities

$$\begin{aligned}\tilde{\mathrm{x}}_{14}{\tilde{\mathrm{x}}_{24}}^{-1}\tilde{\mathrm{x}}_{23}{\tilde{\mathrm{x}}_{13}}^{-1} = {}& u\,\tilde{\mathrm{x}}_{13}{\tilde{\mathrm{x}}_{43}}^{-1}\tilde{\mathrm{x}}_{42}{\tilde{\mathrm{x}}_{12}}^{-1} + v - u\,,\\ v\,\tilde{\mathrm{x}}_{13}{\tilde{\mathrm{x}}_{23}}^{-1}\tilde{\mathrm{x}}_{24}{\tilde{\mathrm{x}}_{14}}^{-1} = {}& -u\,\tilde{\mathrm{x}}_{13}{\tilde{\mathrm{x}}_{43}}^{-1}\tilde{\mathrm{x}}_{42}{\tilde{\mathrm{x}}_{12}}^{-1} + 1\,,\end{aligned} \tag{3.16}$$

we find the following relations necessary to satisfy the superconformal Ward identity, to cancel terms involving ${\Lambda_{1(23)}}^2 \Lambda_{1(24)} \tilde{\mathrm{x}}_{14} {\tilde{\mathrm{x}}_{24}}^{-1} \bar{\epsilon}_2$,

$$\big(q_4 + v\partial_v + (u-1)\partial_u\big) f_{123} + u^{q_1+q_2-2} v^{q_2+q_3-1} \big(q_2 + v\partial_v + u\partial_u\big) f_{134} - u^{q_1+q_2-2} q_1 f_{234} = 0\,, \tag{3.17}$$

to cancel ${\Lambda_{1(23)}}^2 \Lambda_{1(24)} \tilde{\mathrm{x}}_{13} {\tilde{\mathrm{x}}_{23}}^{-1} \bar{\epsilon}_2$,

$$\partial_u f_{123} - u^{q_1+q_2-2} v^{q_2+q_3-1} \partial_v f_{134} = 0\,, \tag{3.18}$$

to cancel ${\Lambda_{1(24)}}^2 \Lambda_{1(23)} \tilde{\mathrm{x}}_{14} {\tilde{\mathrm{x}}_{24}}^{-1} \bar{\epsilon}_2$,

$$\partial_u f_{124} + u^{q_1+q_2-2} \big(q_2 + v\partial_v + u\partial_u\big) f_{134} = 0\,, \tag{3.19}$$

to cancel ${\Lambda_{1(24)}}^2 \Lambda_{1(23)} \tilde{\mathrm{x}}_{13} {\tilde{\mathrm{x}}_{23}}^{-1} \bar{\epsilon}_2$,

$$(\partial_v + \partial_u) f_{124} + u^{q_1+q_2-2} \partial_v f_{134} + u^{q_1+q_2-2} v^{q_1+q_4-2} q_1 f_{234} = 0\,, \tag{3.20}$$

to cancel ${\Lambda_{1(23)}}^2 \Lambda_{1(24)} \tilde{\mathrm{x}}_{13} {\tilde{\mathrm{x}}_{43}}^{-1} \tilde{\mathrm{x}}_{42} {\tilde{\mathrm{x}}_{12}}^{-1} \bar{\epsilon}_1$,

$$\partial_u f_{123} + u^{q_1+q_2-2} \big(q_1 + v\partial_v + u\partial_u\big) f_{234} = 0\,, \tag{3.21}$$

to cancel ${\Lambda_{1(24)}}^2 \Lambda_{1(23)} \tilde{\mathrm{x}}_{13} {\tilde{\mathrm{x}}_{43}}^{-1} \tilde{\mathrm{x}}_{42} {\tilde{\mathrm{x}}_{12}}^{-1} \bar{\epsilon}_1$,

$$-\partial_u f_{124} + u^{q_1+q_2-2} v^{q_1+q_4-1} \partial_v f_{234} = 0\,, \tag{3.22}$$

to cancel ${\Lambda_{1(23)}}^2 \Lambda_{1(24)} \bar{\epsilon}_1$,

$$(v\partial_v + u\partial_u) f_{123} + u^{q_1+q_2-2} v^{q_2+q_3-1} q_2 f_{134} - u^{q_1+q_2-2} \big((v-u)(q_1 + v\partial_v + u\partial_u) - v\partial_v\big) f_{234} = 0\,, \tag{3.23}$$

and finally to remove ${\Lambda_{1(24)}}^2 \Lambda_{1(23)} \bar{\epsilon}_1$,

$$\big(q_3 + v\partial_v + u\partial_u\big) f_{124} - u^{q_1+q_2-2} q_2 f_{134} + u^{q_1+q_2-2} v^{q_1+q_4-1} \big(q_1 + (v-1)\partial_v + u\partial_u\big) f_{234} = 0\,. \tag{3.24}$$

By taking linear combinations the above eight conditions, (3.17),...,(3.24), may be expressed more succinctly as

$$\begin{aligned}
\partial_u f_{123} &= u^{q_1+q_2-2} v^{q_2+q_3-1} \partial_v f_{134}\,, && (3.25a)\\
&= -\, u^{q_1+q_2-2} (q_1 + v\partial_v + u\partial_u) f_{234}\,, && (3.25b)\\
\partial_v f_{123} &= u^{q_1+q_2-1} v^{q_2+q_3-2} \partial_u f_{134}\,, && (3.25c)\\
\partial_u f_{124} &= u^{q_1+q_2-2} v^{q_1+q_4-1} \partial_v f_{234}\,, && (3.25d)\\
&= -\, u^{q_1+q_2-2} (q_2 + v\partial_v + u\partial_u) f_{134}\,, && (3.25e)\\
\partial_v f_{124} &= u^{q_1+q_2-1} v^{q_1+q_4-2} \partial_u f_{234}\,, && (3.25f)\\
(q_4 + v\partial_v + u\partial_u) f_{123} &= -\, u^{q_1+q_2-1} \partial_u f_{234}\,, && (3.25g)\\
(q_3 + v\partial_v + u\partial_u) f_{124} &= -\, u^{q_1+q_2-1} \partial_u f_{134}\,. && (3.25h)
\end{aligned}$$

An important consistency check is that this set of equations are invariant under simultaneous permutations of z_{i+} and q_i. For the cyclic permutation $z_{1+} \to z_{2+} \to z_{3+} \to z_{4+} \to z_{1+}$, which implies $u \leftrightarrow v$, and also letting $q_1 \to q_2 \to q_3 \to q_4 \to q_1$ then the equations are invariant for $f_{123} \to f_{234} \to f_{134} \to f_{124} \to f_{123}$ as expected from the form of (3.9). Similarly under $z_{1+} \leftrightarrow z_{2+}$, when $u \to u' = u/v$, $v \to v' = 1/v$ so that $u\partial_u = u'\partial_{u'}$, $v\partial_v = -v'\partial_{v'} - u'\partial_{u'}$, then also taking $q_1 \leftrightarrow q_2$ and letting $f_{123}(u,v) \to v'^{q_4} f_{123}(u',v')$, $f_{124}(u,v) \to v'^{q_3} f_{124}(u',v')$, $f_{134}(u,v) \to v'^{q_1} f_{234}(u',v')$, $f_{234}(u,v) \to v'^{q_2} f_{134}(u',v')$, we have (3.25$a$) $\leftrightarrow$ (3.25b), (3.25c) $\leftrightarrow$ (3.25g), (3.25d) $\leftrightarrow$ (3.25e) and (3.25f) $\leftrightarrow$ (3.25h). It is straightfoward to obtain equations for f_{123} etc. alone. Eliminating f_{234} from (3.25b) and (3.25g) gives

$$\begin{aligned}&\big\{(u^2-u){\partial_u}^2 + 2uv\partial_u\partial_v + v^2{\partial_v}^2 \\ &\quad + (3-q_2+q_4)(v\partial_v + u\partial_u) - (q_3+q_4-1)\partial_u + (2-q_2)q_4\big\} f_{123} = 0\,,\end{aligned} \tag{3.26}$$

while from (3.25a) and (3.25c) we have

$$\big\{u{\partial_u}^2 - v{\partial_v}^2 + (q_3+q_4-1)\partial_u - (q_1+q_4-1)\partial_v\big\} f_{123} = 0\,. \tag{3.27}$$

Similar equations are easily found for $f_{124}, f_{134}, f_{234}$.[3]

Eqs. (3.26) and (3.27) are identical with a particular generalisation of the differential equation defining the hypergeometric function to two variables [10],

$$\begin{aligned}&x(1-x)f_{xx} - y^2 f_{yy} - 2xyf_{xy} \\ &\quad + \big(\gamma - (\alpha+\beta+1)x\big)f_x - (\alpha+\beta+1)yf_y - \alpha\beta f = 0\,,\end{aligned} \tag{3.28a}$$

$$\begin{aligned}&y(1-y)f_{yy} - x^2 f_{xx} - 2xyf_{xy} \\ &\quad + \big(\gamma' - (\alpha+\beta+1)y\big)f_y - (\alpha+\beta+1)xf_x - \alpha\beta f = 0\,,\end{aligned} \tag{3.28b}$$

since (3.26) has the same form as (3.28a) and (3.27) corresponds exactly to the difference of (3.28a) and (3.28b). These equations have four independent solutions,

$$\begin{aligned}&F_4(\alpha,\beta,\gamma,\gamma';x,y)\,, \qquad x^{1-\gamma}F_4(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,\gamma';x,y)\,, \\ &y^{1-\gamma'}F_4(\alpha+1-\gamma',\beta+1-\gamma',\gamma,2-\gamma';x,y)\,, \\ &x^{1-\gamma}y^{1-\gamma'}F_4(\alpha+2-\gamma-\gamma',\beta+2-\gamma-\gamma',2-\gamma,2-\gamma';x,y)\,,\end{aligned} \tag{3.29}$$

[3] To compare the results here with those of Pickering and West in [6] which are expressed in terms of functions f,g,k,l then $u^{q_3+q_4}v^{q_1+q_4-1}f_{123} = f$, $u^{q_3+q_4}f_{124} = k$, $f_{134} = l$ and $v^{q_1+q_4-1}f_{234} = g$ and we should let $q_i \to \frac{1}{3}q_i$. It is difficult to compare our set of eight equations with theirs although their eqs. (100) and (98) are identical to the equations corresponding to (3.26) and (3.27) for f_{134}, and we have verified that their equations (89,90,91,92) transform appropriately under cyclic permutations using the above relations for f,g,k,l.

with F_4, introduced by Appell in 1880[4], defined by

$$F_4(\alpha,\beta,\gamma,\gamma';x,y) = F_4(\beta,\alpha,\gamma,\gamma';x,y) = \sum_{m,n=0} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!n!\,(\gamma)_m(\gamma')_n}\, x^m y^n \,, \tag{3.30}$$

where

$$(\gamma)_m = \frac{\Gamma(\gamma+m)}{\Gamma(\gamma)} \,. \tag{3.31}$$

The series in (3.30) is convergent for $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1$. The functions F_4 have two crucial, for later use, symmetry properties,

$$F_4(\alpha,\beta,\gamma,\gamma';x,y) = F_4(\alpha,\beta,\gamma',\gamma;y,x) \,, \tag{3.32}$$

which follows trivially from (3.30), and also, under analytic continuation,

$$\begin{aligned} F_4(\alpha,\beta,\gamma,\gamma';x,y) = {}& \frac{\Gamma(\gamma')\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\Gamma(\beta)}\,(-y)^{-\alpha} F_4(\alpha,\alpha+1-\gamma',\gamma,\alpha+1-\beta;x/y,1/y) \\ & + \frac{\Gamma(\gamma')\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\Gamma(\alpha)}\,(-y)^{-\beta} F_4(\beta+1-\gamma',\beta,\gamma,\beta+1-\alpha;x/y,1/y) \,. \end{aligned} \tag{3.33}$$

With the aid of the solutions of (3.28a,b) given by (3.29) the general solution of (3.26) and (3.27) may then be written as

$$\begin{aligned} f_{123}(u,v) = {}& a\, F_4(q_4, 2-q_2, q_3+q_4-1, q_1+q_4-1; u,v) \\ & + b\, u^{q_1+q_2-1} F_4(2-q_3, q_1+1, q_1+q_2, q_1+q_4-1; u,v) \\ & + c\, v^{q_2+q_3-1} F_4(2-q_1, q_3+1, q_3+q_4-1, q_2+q_3; u,v) \\ & + d\, u^{q_1+q_2-1} v^{q_2+q_3-1} F_4(q_2+1, 3-q_4, q_1+q_2, q_2+q_3; u,v) \,, \end{aligned} \tag{3.34}$$

where we have used (3.2) extensively. The complete solution of (3.25$a, ..., h$) is then given by (3.34) along with

$$\begin{aligned} f_{124}(u,v) = {}& -a\, \frac{q_4}{q_1+q_4-1}\, v^{q_1+q_4-1} F_4(q_4+1, 2-q_2, q_3+q_4-1, q_1+q_4; u,v) \\ & - b\, \frac{2-q_3}{q_1+q_4-1}\, u^{q_1+q_2-1} v^{q_1+q_4-1} F_4(3-q_3, q_1+1, q_1+q_2, q_1+q_4; u,v) \\ & - c\, \frac{1}{q_3}(q_2+q_3-1)\, F_4(2-q_1, q_3, q_3+q_4-1, q_2+q_3-1; u,v) \\ & - d\, \frac{q_2+q_3-1}{2-q_4}\, u^{q_1+q_2-1} F_4(q_2+1, 2-q_4, q_1+q_2, q_2+q_3-1; u,v) \,, \end{aligned} \tag{3.35}$$

[4] For historical references see [11].

and

$$
\begin{aligned}
f_{134}(u,v) = a\, & \frac{q_4(2-q_2)}{(q_3+q_4-1)(q_1+q_4-1)}\, u^{q_3+q_4-1} v^{q_1+q_4-1} F_4(q_4+1, 3-q_2, q_3+q_4, q_1+q_4; u,v) \\
& + b\, \frac{q_1+q_2-1}{q_1+q_4-1}\, v^{q_1+q_4-1} F_4(2-q_3, q_1+1, q_1+q_2-1, q_1+q_4; u,v) \\
& + c\, \frac{q_2+q_3-1}{q_3+q_4-1}\, u^{q_3+q_4-1} F_4(2-q_1, q_3+1, q_3+q_4, q_2+q_3-1; u,v) \\
& + d\, \frac{(q_1+q_2-1)(q_2+q_3-1)}{q_2(2-q_4)}\, F_4(q_2, 2-q_4, q_1+q_2-1, q_2+q_3-1; u,v)\,, \qquad (3.36)
\end{aligned}
$$

and

$$
\begin{aligned}
f_{234}(u,v) = & -a\, \frac{q_4}{q_3+q_4-1}\, u^{q_3+q_4-1} F_4(q_4+1, 2-q_2, q_3+q_4, q_1+q_4-1; u,v) \\
& - b\, \frac{1}{q_1}(q_1+q_2-1)\, F_4(2-q_3, q_1, q_1+q_2-1, q_1+q_4-1; u,v) \\
& - c\, \frac{2-q_1}{q_3+q_4-1}\, u^{q_3+q_4-1} v^{q_2+q_3-1} F_4(3-q_1, q_3+1, q_3+q_4, q_2+q_3; u,v) \\
& - d\, \frac{q_1+q_2-1}{2-q_4}\, v^{q_2+q_3-1} F_4(q_2+1, 2-q_4, q_1+q_2-1, q_2+q_3; u,v)\,. \qquad (3.37)
\end{aligned}
$$

In each case the F_4 functions, defined by (3.30), satisfy $\alpha+\beta=\gamma+\gamma'+1$.[5]

4. Crossing Symmetry Relations

Although the above results provide a complete solution of the superconformal Ward identities, the solutions of the differential equations are further constrained by considering permutations of z_{i+} and simultaneously q_i. Such permutations act on the invariants u, v so that in general they are not restricted to the domain of convergence of the F_4 functions and thus analytic continuation is necessary. It is essential that the results be independent of the path of analytic continuation or that they be monodromy invariant.

Firstly for cyclic permutations, when $u \leftrightarrow v$, in (3.9) we require

$$
\begin{aligned}
f_{123}(q_1,q_2,q_3,q_4;u,v) &= f_{124}(q_2,q_3,q_4,q_1;v,u) \\
&= f_{134}(q_3,q_4,q_1,q_2;u,v) = f_{234}(q_4,q_1,q_2,q_3;v,u)\,,
\end{aligned}
\qquad (4.1)
$$

[5] For particular values of q_i the infinite series for F_4 truncates. Thus for $q_2=0$ from the terms $\propto d$ we have $f_{134}=$ const. while $f_{123}=f_{124}=f_{234}=0$, which corresponds to the situation when the four point function reduces to a three point function. If $q_1=-1$ there is a solution $f_{123}=bu^{q_2-2}$, $f_{124}=b\frac{q_3-2}{q_4-2}\,u^{q_2-2}v^{q_4-2}$, $f_{134}=b\frac{q_2-2}{q_4-2}\,v^{q_4-2}$, $f_{234}=b(q_2-2)(1+\frac{q_3-2}{q_2-2}u+\frac{q_3-2}{q_4-2}v)$. For $q_2=0$ as well this coincides, up to misprints, with the solution (104) in [6].

which, with (3.32), leads to

$$
\begin{aligned}
a(q_1,q_2,q_3,q_4) &= -\frac{1}{q_4}(q_3+q_4-1)c(q_2,q_3,q_4,q_1)\,, && (4.2a)\\
d(q_1,q_2,q_3,q_4) &= -\frac{2-q_4}{q_1+q_2-1}\,b(q_2,q_3,q_4,q_1)\,, && (4.2b)\\
b(q_1,q_2,q_3,q_4) &= -\frac{q_1}{q_1+q_2-1}\,a(q_2,q_3,q_4,q_1)\,, && (4.2c)\\
c(q_1,q_2,q_3,q_4) &= -\frac{q_3+q_4-1}{2-q_1}\,d(q_2,q_3,q_4,q_1)\,. && (4.2d)
\end{aligned}
$$

The critical conditions arise from considering $z_{1+}, q_1 \leftrightarrow z_{2+}, q_2$ so that $u \to u' = u/v$, $v \to v' = 1/v$. In this case we require

$$
\begin{aligned}
f_{123}(q_1,q_2,q_3,q_4;u,v) &= v'^{q_4} f_{123}(q_2,q_1,q_3,q_4;u',v')\,, && (4.3a)\\
f_{124}(q_1,q_2,q_3,q_4;u,v) &= v'^{q_3} f_{124}(q_2,q_1,q_3,q_4;u',v')\,, && (4.3b)\\
f_{134}(q_1,q_2,q_3,q_4;u,v) &= v'^{q_1} f_{234}(q_2,q_1,q_3,q_4;u',v')\,, && (4.3c)
\end{aligned}
$$

Using now (3.33), we get from (4.3a), for one path of analytic continuation,

$$
\begin{aligned}
&\left\{a(q_1,q_2,q_3,q_4)\frac{e^{-i\pi q_4}\Gamma(q_1+q_4-1)}{\Gamma(q_1-1)\Gamma(2-q_2)} + c(q_1,q_2,q_3,q_4)\frac{e^{i\pi(q_1-2)}\Gamma(q_2+q_3)}{\Gamma(1-q_4)\Gamma(q_3+1)}\right\}\Gamma(q_1+q_3-1)\\
&\qquad = a(q_2,q_1,q_3,q_4)\,, \qquad (4.4a)\\
&\left\{a(q_1,q_2,q_3,q_4)\frac{e^{i\pi(q_2-1)}\Gamma(q_1+q_4-1)}{\Gamma(-q_3)\Gamma(q_4)} + c(q_1,q_2,q_3,q_4)\frac{e^{-i\pi(q_3+1)}\Gamma(q_2+q_3)}{\Gamma(q_2-1)\Gamma(2-q_1)}\right\}\Gamma(q_2+q_4-2)\\
&\qquad = c(q_2,q_1,q_3,q_4)\,, \qquad (4.4b)\\
&\left\{b(q_1,q_2,q_3,q_4)\frac{e^{i\pi(q_3-2)}\Gamma(q_1+q_4-1)}{\Gamma(-q_2)\Gamma(q_1+1)} + d(q_1,q_2,q_3,q_4)\frac{e^{-i\pi(q_2+1)}\Gamma(q_2+q_3)}{\Gamma(q_3-1)\Gamma(3-q_4)}\right\}\Gamma(q_1+q_3-1)\\
&\qquad = b(q_2,q_1,q_3,q_4)\,, \qquad (4.4c)\\
&\left\{b(q_1,q_2,q_3,q_4)\frac{e^{-\pi(q_1+1)}\Gamma(q_1+q_4-1)}{\Gamma(q_4-2)\Gamma(2-q_3)} + d(q_1,q_2,q_3,q_4)\frac{e^{i\pi(q_4-3)}\Gamma(q_2+q_3)}{\Gamma(-q_1)\Gamma(q_2+1)}\right\}\Gamma(q_2+q_4-2)\\
&\qquad = d(q_2,q_1,q_3,q_4)\,. \qquad (4.4d)
\end{aligned}
$$

For consistency the imaginary parts of the left hand sides of (4.4a, .., d) must vanish. In this case the same result is obtained for the other possible analytic continuation since their difference is just in the imaginary part.[6] From (4.4a) or (4.4c), using $\Gamma(x)\Gamma(1-x) = -\Gamma(x+1)\Gamma(-x) = \pi/\sin\pi x$, we get

$$
c(q_1,q_2,q_3,q_4) = -\frac{\Gamma(q_1+q_4-1)\Gamma(2-q_1)\Gamma(q_3+1)}{\Gamma(q_2+q_3)\Gamma(2-q_2)\Gamma(q_4)}\,a(q_1,q_2,q_3,q_4)\,, \qquad (4.5)
$$

[6] For a related discussion in two dimensions see [12].

while from (4.4c) or (4.4d) we also obtain

$$d(q_1,q_2,q_3,q_4) = -\frac{\Gamma(q_1+q_4-1)\Gamma(3-q_4)\Gamma(q_1+1)}{\Gamma(q_2+q_3)\Gamma(2-q_3)\Gamma(q_1+1)}\, b(q_1,q_2,q_3,q_4)\,. \tag{4.6}$$

Using (4.5), (4.4a) reduces to

$$a(q_1,q_2,q_3,q_4)\,\frac{\Gamma(q_1+q_3-1)\Gamma(2-q_1)}{\Gamma(q_2+q_3-1)\Gamma(2-q_2)} = a(q_2,q_1,q_3,q_4)\,, \tag{4.7}$$

while (4.4b) gives an equivalent, by virtue of (4.5), relation for c. Similarly (4.4c) or (4.4d) give

$$b(q_1,q_2,q_3,q_4)\,\frac{\Gamma(q_1+q_3-1)\Gamma(q_2+1)}{\Gamma(q_2+q_3-1)\Gamma(q_1+1)} = b(q_2,q_1,q_3,q_4)\,. \tag{4.8}$$

Identical results may also be obtained from (4.3b) or (4.3c).

To simplify further, it is convenient to define

$$\begin{aligned} a(q_1,q_2,q_3,q_4) &= \Gamma(q_2+q_3-1)\Gamma(q_1+q_2-1)\Gamma(2-q_2)\Gamma(q_4)\, A(q_1,q_2,q_3,q_4)\,,\\ c(q_1,q_2,q_3,q_4) &= \Gamma(q_1+q_4-2)\Gamma(q_1+q_2-1)\Gamma(2-q_1)\Gamma(q_3+1)\, A(q_1,q_2,q_3,q_4)\,, \end{aligned} \tag{4.9}$$

and also

$$\begin{aligned} b(q_1,q_2,q_3,q_4) &= \Gamma(q_2+q_3-1)\Gamma(q_3+q_4-2)\Gamma(2-q_3)\Gamma(q_1+1)\, B(q_1,q_2,q_3,q_4)\,,\\ d(q_1,q_2,q_3,q_4) &= \Gamma(q_1+q_4-2)\Gamma(q_3+q_4-2)\Gamma(3-q_4)\Gamma(q_2+1)\, B(q_1,q_2,q_3,q_4)\,, \end{aligned} \tag{4.10}$$

where the second lines follow from (4.5) and (4.6) respectively. Now (4.7) and (4.8) become

$$A(q_1,q_2,q_3,q_4) = A(q_2,q_1,q_3,q_4)\,, \qquad B(q_1,q_2,q_3,q_4) = B(q_2,q_1,q_3,q_4)\,. \tag{4.11}$$

Inserting (4.9) into (4.2a) and (4.10) into (4.2b) gives

$$A(q_1,q_2,q_3,q_4) = A(q_2,q_3,q_4,q_1)\,, \qquad B(q_1,q_2,q_3,q_4) = B(q_2,q_3,q_4,q_1)\,. \tag{4.12}$$

Similarly (4.2c) and (4.2d) lead to

$$B(q_1,q_2,q_3,q_4) = A(q_2,q_3,q_4,q_1)\,, \qquad A(q_1,q_2,q_3,q_4) = B(q_2,q_3,q_4,q_1)\,. \tag{4.13}$$

In consequence we must have $B = A$ so that the chiral four point function is determined up to a single overall constant. $f_{123}(u,v)$ is given by substituting (4.9) and (4.10), with $A = B$, into (3.34) while $f_{124}(u,v)$, $f_{134}(u,v)$, $f_{234}(u,v)$ are given by similar expressions which may easily be found from (4.1).

For completeness we may also consider the permutation $z_{1+}, q_1 \leftrightarrow z_{3+}, q_3$ and $z_{2+}, q_2 \leftrightarrow z_{4+}, q_4$, when u, v are invariant. This requires $f_{123}(q_3,q_4,q_1,q_2;u,v) = f_{134}(q_1,q_2,q_3,q_4;u,v)$ and $f_{124}(q_3,q_4,q_1,q_2;u,v) = f_{234}(q_1,q_2,q_3,q_4;u,v)$. This relates a to d and b to d and with (4.9) and (4.10) would require $A(q_3,q_4,q_1,q_2) = B(q_1,q_2,q_3,q_4)$ and $B(q_3,q_4,q_1,q_2) = A(q_1,q_2,q_3,q_4)$.

5. Short Distance Expansions

In the limits $x_{1+} \to x_{2+}$ or $x_{3+} \to x_{4+}$ we have, with the definitions in (2.11), $u \to 0$, $v \to 1$. Similarly for $x_{1+} \to x_{4+}$ or $x_{2+} \to x_{3+}$, $u \to 1$, $v \to 0$ while for $x_{1+} \to x_{3+}$ or $x_{2+} \to x_{4+}$, $1/v \to 0$, $u/v \to 1$. In order to understand the behaviour in these short distance limits a different expansion, which reveals the form of the F_4 function for one of the arguments near the singular point 1, than that given by (3.30) is necessary.

In the simpler case of standard hypergeometric functions the relevant results are easily obtained. The associated second order ordinary differential equation has three singular points at $0, 1, \infty$. The two independent solutions may be taken as $F(\alpha, \beta; \gamma; x)$ and $x^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma; x)$, which are thus given as an expansion in powers of x, but they can be equally expressed in terms of $F(\alpha, \beta; \alpha + \beta + 1 - \gamma; 1 - x)$ and $(1 - x)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - x)$, determining the form as $x \to 1$, or $(-x)^{-\alpha} F(\alpha, \alpha + 1 - \gamma; \alpha + 1 - \beta; 1/x)$ and $(-x)^{-\beta} F(\beta + 1 - \gamma, \beta; \beta + 1 - \alpha; 1/x)$, which gives an equivalent expression in terms of $1/x$, revealing the behaviour at ∞. Any of these functions may be written in terms of a linear combination of the two functions defined by a series expansion at either of the other singular points. For the F_4 function, given by (3.30), (3.33) gives an equivalent result involving the behaviour for one variable at ∞. To determine an analogous result for the form of F_4, as used here, for one argument approaching 1 we define the new function

$$G(\alpha, \beta, \gamma, \delta; x, y) = \sum_{m,n=0} \frac{(\delta - \alpha)_m (\delta - \beta)_m}{m! (\gamma)_m} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{n! (\delta)_{2m+n}} x^m y^n \,. \tag{5.1}$$

Writing the sum over n in terms of a hypergeometric function and using the identity relating hypergeometric functions of arguments y and $1 - y$ we may obtain[7]

$$\begin{aligned} G(\alpha, \beta, \gamma, \delta; x, 1 - y) = {} & \frac{\Gamma(\delta)\Gamma(\delta - \alpha - \beta)}{\Gamma(\delta - \alpha)\Gamma(\delta - \beta)} F_4(\alpha, \beta, \gamma, \alpha + \beta + 1 - \delta; x, y) \\ & + \frac{\Gamma(\delta)\Gamma(\alpha + \beta - \delta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\delta-\alpha-\beta} F_4(\delta - \alpha, \delta - \beta, \gamma, \delta - \alpha - \beta + 1; x, y) \,. \end{aligned} \tag{5.2}$$

From this it is evident that

$$G(\alpha, \beta, \gamma, \delta; x, 1 - y) = y^{\delta-\alpha-\beta} G\big(\delta - \alpha, \delta - \beta, \gamma, \delta; x, 1 - y\big) \,, \tag{5.3}$$

and using (3.33) and standard Γ-function identities we find, from (5.2),

$$G(\alpha, \beta, \gamma, \delta; x, 1 - y) = y^{-\alpha} G\big(\alpha, \delta - \beta, \gamma, \delta; x/y, 1 - 1/y\big) \,. \tag{5.4}$$

[7] The behaviour of F_4 functions near $y = 1$ was investigated quite recently by Exton [13]. The function G is essentially that defined by Exton and the relation obtained here is a special case of his formulae. The series for G in (5.1) also features in a related discussion in [14].

Using (5.2) with (3.34) where a, b, c, d are given by (4.9) and (4.10) we may now find in terms of the function G,

$$\begin{aligned} f_{123}(u,v) = {} & \frac{\Gamma(q_1+q_2-1)\Gamma(2-q_1)\Gamma(2-q_2)\Gamma(q_3+1)\Gamma(q_4)}{\Gamma(q_3+q_4+1)}\, A \\ & \times G(q_4, 2-q_2, q_3+q_4-1, q_3+q_4+1; u, 1-v) \\ & + \frac{\Gamma(q_3+q_4-2)\Gamma(q_1+1)\Gamma(q_2+1)\Gamma(2-q_3)\Gamma(3-q_4)}{\Gamma(q_1+q_2+2)}\, B \\ & \times u^{q_1+q_2-1} G(2-q_3, q_1+1, q_1+q_2, q_1+q_2+2; u, 1-v)\,, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} f_{124}(u,v) = {} & \frac{\Gamma(q_1+q_2-1)\Gamma(2-q_1)\Gamma(2-q_2)\Gamma(q_3)\Gamma(q_4+1)}{\Gamma(q_3+q_4+1)}\, A \\ & \times G(q_3, 2-q_1, q_3+q_4-1, q_3+q_4+1; u, 1-v) \\ & + \frac{\Gamma(q_3+q_4-2)\Gamma(q_1+1)\Gamma(q_2+1)\Gamma(3-q_3)\Gamma(2-q_4)}{\Gamma(q_1+q_2+2)}\, B \\ & \times u^{q_1+q_2-1} G(2-q_4, q_2+1, q_1+q_2, q_1+q_2+2; u, 1-v)\,, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} f_{134}(u,v) = {} & \frac{\Gamma(q_3+q_4-1)\Gamma(q_1+1)\Gamma(q_2)\Gamma(2-q_3)\Gamma(2-q_4)}{\Gamma(q_1+q_2+1)}\, B \\ & \times G(q_2, 2-q_4, q_1+q_2-1, q_1+q_2+1; u, 1-v) \\ & + \frac{\Gamma(q_1+q_2-2)\Gamma(2-q_1)\Gamma(3-q_2)\Gamma(q_3+1)\Gamma(q_4+1)}{\Gamma(q_3+q_4+2)}\, A \\ & \times u^{q_3+q_4-1} G(2-q_1, q_3+1, q_3+q_4, q_3+q_4+2; u, 1-v)\,, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} f_{234}(u,v) = {} & \frac{\Gamma(q_3+q_4-1)\Gamma(q_1)\Gamma(q_2+1)\Gamma(2-q_3)\Gamma(2-q_4)}{\Gamma(q_1+q_2+1)}\, B \\ & \times G(q_1, 2-q_3, q_1+q_2-1, q_1+q_2+1; u, 1-v) \\ & + \frac{\Gamma(q_1+q_2-2)\Gamma(3-q_1)\Gamma(2-q_2)\Gamma(q_3+1)\Gamma(q_4+1)}{\Gamma(q_3+q_4+2)}\, A \\ & \times u^{q_3+q_4-1} G(2-q_2, q_4+1, q_3+q_4, q_3+q_4+2; u, 1-v)\,. \end{aligned} \tag{5.8}$$

With these expressions the relations ($4.3a, b, c$) are easy to verify using (5.3) and (5.4). Clearly the results are manifestly analytic at $v = 1$.

In general if we define

$$\begin{aligned} H(\alpha,\beta,\gamma,\delta;u,v) = {} & \frac{\Gamma(1-\gamma)}{\Gamma(\delta)}\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha)\Gamma(\delta-\beta)\, G(\alpha,\beta,\gamma,\delta;u,1-v) \\ & + \frac{\Gamma(\gamma-1)}{\Gamma(\delta-2\gamma+2)}\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)\Gamma(\delta-\gamma-\alpha+1)\Gamma(\delta-\gamma-\beta+1) \\ & \times u^{1-\gamma} G(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, \delta-2\gamma+2; u, 1-v)\,, \end{aligned} \tag{5.9}$$

then, from (5.2) and (3.32), we have

$$H(\alpha,\beta,\gamma,\delta;u,v) = H(\alpha,\beta,\alpha+\beta+1-\delta,\alpha+\beta+1-\gamma;v,u)\,. \tag{5.10}$$

The expression given by (5.9) coincides, for appropriate choices of $\alpha,\beta,\gamma,\delta$, with the forms in (5.5),(5.6),(5.7),(5.8) if $A=B$. From (5.4) we have

$$H(\alpha,\beta,\gamma,\delta;u,v) = v^{-\alpha} H\big(\alpha,\delta-\beta,\gamma,\delta;u/v,1/v\big)\,, \tag{5.11}$$

and from (5.3) together with (5.10) we may obtain

$$\begin{aligned} H(\alpha,\beta,\gamma,\delta;u,v) &= v^{\delta-\alpha-\beta} H\big(\delta-\alpha,\delta-\beta,\gamma,\delta;u,v\big) \\ &= u^{1-\gamma} v^{\delta-\alpha-\beta} H\big(\delta-\alpha+1-\gamma,\delta-\beta+1-\gamma,2-\gamma,\delta-2\gamma+2;u,v\big)\,. \end{aligned} \tag{5.12}$$

Together (5.10), (5.11) and (5.12) are sufficient to obtain the necessary symmetry relations.

If $\gamma=1$ the definition (5.9) reduces to

$$\begin{aligned} H(\alpha,\beta,1,\delta;u,v) = {}& \frac{1}{\Gamma(\delta)}\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha)\Gamma(\delta-\beta)\bigg\{-\ln u\, G(\alpha,\beta,1,\delta;u,1-v) \\ &+ \sum_{m,n=0} \frac{(\delta-\alpha)_m(\delta-\beta)_m}{(m!)^2}\,\frac{(\alpha)_{m+n}(\beta)_{m+n}}{n!(\delta)_{2m+n}}\, f_{mn}\, u^m(1-v)^n\bigg\}\,, \\ f_{mn} = {}& 2\psi(1+m)+2\psi(\delta+2m+n)-\psi(\delta-\alpha+m)-\psi(\delta-\beta+m) \\ &-\psi(\alpha+m+n)-\psi(\beta+m+n)\,, \end{aligned} \tag{5.13}$$

involving $\ln u$ although (5.10), (5.11) and (5.12) remain valid.

6. Counting for higher N-point Functions

The constraints arising from superconformal invariance uniquely determine the form of the scalar chiral superfield four point function, as shown in section 3. We attempt here to count the number of independent functions of the appropriate generalisations of u,v and to determine the number of independent differential equations constraining them for higher point functions of scalar chiral superfields.

In order to undertake such an analysis the superconformal group is restricted by first using superconformal transformations to set

$$x_{1+}=0\,,\quad x_{2+}=\infty\,,\quad \theta_1=0\,,\quad \theta_2=0\,. \tag{6.1}$$

The residual symmetry group is then that generated by matrices of the form (2.2) with $a, b, \epsilon, \bar{\eta}$ zero. The associated subgroup of G_0 is given by rotations and scale, $U(1)_R$, transformations for which the infinitesimal parameters are $\omega^{ab}, \kappa, \bar{\kappa}$. With (6.1) we have

$$\Lambda_{1(2i)} = \theta_i \tilde{\mathrm{x}}_i{}^{-1} \,. \tag{6.2}$$

After imposing (6.1) an N-point function for chiral superfields depends only on $z_{i+} = (x_{i+}, \theta_i)$, $i = 3, \dots N$. The rotational and scale invariant variables constructed from z_{i+} are then easily seen to be

$$u_{ij} = \frac{x_{j+}{\cdot}x_{i+}}{x_{3+}{}^2} \,, \quad j \geq i > 3 \,, \qquad u_i = \frac{2x_{i+}{\cdot}x_{3+}}{x_{3+}{}^2} \,, \quad i > 3 \,. \tag{6.3}$$

Evidently there are $\frac{1}{2}(N-3)(N-2)$ such u_{ij} and $N-3$ u_i giving $\frac{1}{2}N(N-3)$ in total. However, since we are concerned only with four dimensional space, if $N \geq 7$ we may express x_{i+}, $i \geq 7$ in terms of x_{j+}, $j = 3,4,5,6$, with coefficients involving $x_{i+}{\cdot}x_{j+}$. This ensures that for the u_{ij} in (6.3) we may therefore restrict to $j = 3,4,5,6$ and the number of such independent invariants becomes $N-3+N-4+N-5$ giving, with the u_i as well, altogether $4N-15$. Clearly for $N \geq 7$ the 15 parameter conformal group $SU(2,2)$ is acting transitively. In fact this result gives the correct number also for $N = 5, 6$ whereas for $N = 4$ we have just two invariants u_{44}, u_4.

For application to N-point functions for chiral superfields with $\sum_i q_i = 3$ we need a set of nilpotent monomials $\mathrm{O}(\theta^2)$, generalising $\Lambda_{i(jk)}{}^2$, $i < j < k$, used for $N = 4$. With the choice in (6.1) we may take as a linearly independent basis

$$\Xi_i = \theta_i{}^2 \,, \quad i \geq 3 \,, \qquad \Xi_{ij,r} = \theta_j M_r \tilde{\theta}_i \,, \quad j > i \geq 3 \,, \tag{6.4}$$

where $(M_r)_\alpha{}^\beta$ are linearly independent 2×2 matrices constructed from x_{i+}, which we assume to have dimension zero. To achieve the form in (6.4) we may note that $\theta_i M \tilde{\theta}_j = \theta_j \tilde{M} \tilde{\theta}_i$ where $\tilde{M}_\alpha{}^\beta = \epsilon_{\alpha\gamma}\epsilon^{\beta\delta} M_\delta{}^\gamma$. For $N \geq 5$ it is sufficient to take just $r = 1,2,3,4$ since any such matrix may be expressed in terms of the basis formed by 1, $\mathrm{x}_{4+}\tilde{\mathrm{x}}_{3+}/x_{3+}{}^2$, $\mathrm{x}_{5+}\tilde{\mathrm{x}}_{3+}/x_{3+}{}^2$, $\mathrm{x}_{5+}\tilde{\mathrm{x}}_{4+}/x_{3+}{}^2$. Clearly there are $N-2$ independent Ξ_i and $2(N-3)(N-2)$ independent $\Xi_{ij,r}$ giving $(2N-5)(N-2)$ independent such Ξ_I in total. The general N-point function can then be written in terms of scalar functions of the invariants u as

$$\big(x_{2+}{}^2\big)^{q_2} \big\langle \phi_1(z_{1+})\phi_2(z_{2+}) \dots \phi_N(z_{N+}) \big\rangle \Big|_{\substack{x_{1+}=0,\theta_1=0 \\ x_{2+}\to\infty,\theta_2=0}} = \big(x_{3+}{}^2\big)^{q_2-2} \sum_I \Xi_I \, f_I(u) \,. \tag{6.5}$$

For $N = 4$ there are only two independent matrices M_r since they can be restricted to the basis formed by 1, $\mathrm{x}_{4+}\tilde{\mathrm{x}}_{3+}/x_{3+}{}^2$. There are therefore just four independent Ξ_I, which

can be alternatively expressed in terms ${\Lambda_{1(23)}}^2, {\Lambda_{1(24)}}^2, {\Lambda_{1(34)}}^2, {\Lambda_{2(34)}}^2$. Thus these results are in accord with the treatment in section 3 and the expansion of the four point function exhibited in (3.9). For $N > 4$ it is easy to see that the Ξ_I cannot all be expressed solely in terms of ${\Lambda_{i(jk)}}^2$, for $N = 5$ there are 10 ${\Lambda_{i(jk)}}^2$, $i < j < k$, whereas the basis in (6.4) gives 15.

The non trivial constraints in the superconformal Ward identities (3.1) arise from the terms involving $\bar{\epsilon}$ or η. From (2.1) such terms are $\mathrm{O}(\theta^3)$, and the number of independent conditions, which involve linear first order partial differential equations for the $f_I(u)$, is equal to the number of independent monomials of the form $\theta^3\bar{\epsilon}$ or $\theta^3\eta$. For $N \geq 4$ we have

$$ {\theta_j}^2\, \theta_i N_s \bar{\epsilon}\,, \qquad {\theta_j}^2\, \theta_i M_r \eta\,, \quad i \neq j\,,\ i,j \geq 3\,, \tag{6.6} $$

where $(N_s)_{\alpha\dot{\alpha}}$ is a linearly independent set of 2×2 matrices. For $N \geq 5$ M_r may be reduced to the basis described above and for $N \geq 6$ N_s may be given in terms of the basis $\mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5, \mathrm{x}_6$. Thus if $N \geq 6$ there are four possible M_r and also four N_s so that the number of independent monomials of the form (6.6) is $8(N-3)(N-2)$. In addition to (6.6) for $N \geq 5$ we may also construct

$$ \theta_k M_r \tilde{\theta}_j\, \theta_i N_s \bar{\epsilon}\,, \qquad \theta_k M_r \tilde{\theta}_j\, \theta_i M_s \eta\,, \quad k > j > i \geq 3\,. \tag{6.7} $$

The ordering $k > j > i$ is achieved by Fierz type identities, since if $i > j$ we may write $\tilde{\theta}_j \theta_i = -\sum_r \theta_i M_r \tilde{\theta}_j M'_r$ for $\{M_r\}$ forming a basis for 2×2 matrices and where M'_r satisfy $\mathrm{tr}(M'_r M_s) = \delta_{rs}$. The expressions (6.6) and (6.7) are a linearly independent basis. Assuming both four linearly independent M_r and N_s we have $\frac{16}{3}(N-4)(N-3)(N-2)$ monomials of the form (6.7). Combining these with those exhibited in (6.6) we have $\frac{8}{3}(N-3)(N-2)(2N-5)$ in total giving this number of constraints on the f_I. Since there are $(N-2)(2N-5)$ f_I depending on $4N-15$ variables the number of functional degrees of freedom remaining after imposing the differential constraints is $\frac{1}{3}(N-2)(2N-5)(4N-21)$, which for $N = 6$ is 28.

When $N = 4$ the relevant monomials are only of the form in (6.6), with $i, j = 3, 4$, and we further restrict to $r = 1, 2$, corresponding to $1, \mathrm{x}_{4+}\tilde{\mathrm{x}}_{3+}/{x_{3+}}^2$, and also $s = 1, 2$, since N_s is also restricted to the basis $\mathrm{x}_3, \mathrm{x}_4$. This gives 8 independent monomials so that there are 8 conditions on the 4 functions of 2 variables f_I. In appendix A we explicitly obtain the 8 relations for f_1, f_2, f_3, f_4 which are equivalent to those found earlier in (3.17), $\ldots$, (3.24) or (3.25a, ..h). For $N = 4$ we therefore expect a unique functional form for the solution, up to choices of integration constants, which of course is in accord with the results of section 3. For $N = 5$ there are apparently 80 constraints on 15 functions f_I of 5 variables leading to an overdetermined system. However if any $q_i = 0$ the equations should reduce to those determining the four point function so that there are possible linear dependencies in the constraints but this case is too complicated to investigate in detail.

7. Operator Product Expansion

In conformal field theories there are further conditions to be obtained by imposing the operator product expansion. A four point function involving fields at x_1, x_2, x_3, x_4 may be expanded as a convergent series for $x_1 \approx x_2,\ x_3 \approx x_4$ in terms of an infinite set of quasi-primary fields and also equivalently for $x_1 \approx x_3,\ x_2 \approx x_4$. The equality of the two expansions provides a constraint on the operator content of the theory. In this section we consider the contribution of a single chiral quasi-primary superfield and its descendents to the four point function of chiral superfields.

The operator product coefficients are determined by the two and three point functions. For scalar fields φ_i, with dimension δ_i in dimension d, then the two point functions in a conformal theory may be chosen as

$$\langle \varphi_i(x_1)\varphi_j(x_2)\rangle = \frac{\delta_{ij}}{{r_{12}}^{\delta_i}} \, , \tag{7.1}$$

and the three point functions are then

$$\langle \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\rangle = \frac{C_{ijk}}{{r_{12}}^{\frac{1}{2}(\delta_i+\delta_j-\delta_k)}\,{r_{23}}^{\frac{1}{2}(\delta_j+\delta_k-\delta_i)}\,{r_{31}}^{\frac{1}{2}(\delta_k+\delta_i-\delta_j)}} \, . \tag{7.2}$$

The contribution of the field φ_k to the operator product of φ_i and φ_j is then determined from (7.1) and (7.2) to be

$$\varphi_i(x_1)\varphi_j(x_2) \sim C_{ijk}\, {r_{12}}^{-\frac{1}{2}(\delta_i+\delta_j-\delta_k)} C^{\frac{1}{2}(\delta_k+\delta_i-\delta_j),\frac{1}{2}(\delta_k-\delta_i+\delta_j)}_{\delta_k+1-\frac{1}{2}d}(x_{12},\partial_{x_2})\varphi_k(x_2) \, , \tag{7.3}$$

where

$$C^{a,b}_{S+1-\frac{1}{2}d}(x_{12},\partial_{x_2})\frac{1}{{r_{23}}^{S}} = \frac{1}{{r_{13}}^{a}{r_{23}}^{b}} \, , \qquad S = a+b \, . \tag{7.4}$$

$C^{a,b}_{\kappa}(s,\partial)$ is given as an infinite series in $s{\cdot}\partial$ and $s^2\partial^2$ in appendix D, $C^{a,b}_{\kappa}(0,\partial) = 1$. For application to four point functions we have

$$\begin{aligned} C^{a,b}_{S+1-\frac{1}{2}d}(x_{12},\partial_{x_2})\frac{1}{{r_{23}}^{e}{r_{24}}^{f}} &= \frac{1}{{r_{14}}^{a}{r_{24}}^{b}}\left(\frac{r_{14}}{r_{13}}\right)^{e} G(b,e,S+1-\tfrac{1}{2}d,S;u,1-v) \\ &= \frac{1}{{r_{13}}^{a}{r_{23}}^{b}}\left(\frac{r_{13}}{r_{14}}\right)^{f} G(a,f,S+1-\tfrac{1}{2}d,S;u,1-v) \, , \end{aligned} \tag{7.5}$$

with $S = a+b = e+f$ and using (5.3).

Similar results may be obtained for chiral scalar superfields, restricting now to $d=4$. The basic two point function involves a chiral superfield ϕ and its anti-chiral conjugate $\bar\phi$ [1],

$$\langle \phi_i(z_{1+})\bar\phi_j(z_{2-})\rangle = \frac{\delta_{ij}}{\big(x_{1+}-2i\theta_1\sigma\bar\theta_2-x_{2-}\big)^{2q_i}} \, , \tag{7.6}$$

and the corresponding three point function has the form

$$\langle \phi_i(z_{1+})\phi_j(z_{2+})\bar{\phi}_k(z_{3-})\rangle = \frac{C_{ij\bar{k}}}{\big(x_{1+} - 2i\theta_1\sigma\bar{\theta}_3 - x_{3-}\big)^{2q_i}\big(x_{2+} - 2i\theta_2\sigma\bar{\theta}_3 - x_{3-}\big)^{2q_j}}\,, \quad (7.7)$$

and is only possible if $q_k = q_i + q_j$. As a consequence of (7.6) and (7.7) the chiral superfield contributes to the operator product expansion of ϕ_i and ϕ_j

$$\phi_i(z_{1+})\phi_j(z_{2+}) \sim C_{ij\bar{k}}\, \mathcal{C}^{q_1,q_2}(z_{12+}, \partial_{z_{2+}})\phi_k(z_{2+})\,, \quad (7.8)$$

where

$$\begin{aligned} &\mathcal{C}^{q_1,q_2}(z_{12+}, \partial_{z_{2+}})\frac{1}{\big(x_{2+} - 2i\theta_2\sigma\bar{\theta} - x_{-}\big)^{2q}} \\ &\quad = \frac{1}{\big(x_{1+} - 2i\theta_1\sigma\bar{\theta} - x_{-}\big)^{2q_1}\big(x_{2+} - 2i\theta_2\sigma\bar{\theta} - x_{-}\big)^{2q_2}}\,, \qquad q = q_1 + q_2\,. \end{aligned} \quad (7.9)$$

As shown in Appendix E we may express $\mathcal{C}^{q_1,q_2}$ as

$$\begin{aligned} \mathcal{C}^{q_1,q_2}(z_{12+}, \partial_{z_{2+}}) = {}& C^{q_1,q_2}_{q-1}(x_{12+}, \partial_{x_{2+}}) + \frac{q_1}{q}\, C^{q_1+1,q_2}_{q-1}(x_{12+}, \partial_{x_{2+}})\theta_{12}\partial_{\theta_2} \\ &+ \frac{q_1 q_2}{2q(q^2-1)}\, C^{q_1+1,q_2+1}_{q}(x_{12+}, \partial_{x_{2+}})\theta_{12}\mathrm{x}_{12+}\tilde{\partial}_{x_{2+}}\partial_{\theta_2} \\ &- \frac{q_1(q_1-1)}{4q(q-1)}\, {\theta_{12}}^2 C^{q_1+1,q_2}_{q}(x_{12+}, \partial_{x_{2+}}){\partial_{\theta_2}}^2\,, \end{aligned} \quad (7.10)$$

where $\tilde{\partial}_x = \tilde{\sigma}\cdot\partial_x$. The result (7.8) has no singular terms for $z_{1+} \to z_{2+}$ so that the chiral superfields form a closed algebraic ring with the scale dimension/R-charge additive.

We may use (7.8) to determine the contribution arising from a chiral scalar superfield ϕ_i to the four point function $\langle\phi_1(z_{1+})\phi_2(z_{2+})\phi_3(z_{3+})\phi_4(z_{4+})\rangle$ arising from the operator product expansion for $\phi_1(z_{1+})\phi_2(z_{2+})$ where $q_i = q = q_1 + q_2 = 3 - q_3 - q_4$. From (3.3)

$$\big\langle \phi_i(z_{2+})\phi_3(z_{3+})\phi_4(z_{4+})\big\rangle = C_{i34}\, {\Lambda_{2(34)}}^2 \frac{{r_{34}}^{q-2}}{{r_{23}}^{1-q_4}{r_{24}}^{1-q_3}}\,. \quad (7.11)$$

and using (7.8) gives, after lengthy calculations extending those which led to (7.5) which are described in appendix E, exactly the form shown in (3.9) with

$$\begin{aligned} f_{123}(u,v) &= -\, C_{12\bar{i}}C_{i34}\, \frac{q_1q_2}{q(q^2-1)}(2-q_4)u^{q-1}G(2-q_3, q_1+1, q, q+2; u, 1-v)\,, \\ f_{124}(u,v) &= -\, C_{12\bar{i}}C_{i34}\, \frac{q_1q_2}{q(q^2-1)}(2-q_3)u^{q-1}G(2-q_4, q_2+1, q, q+2; u, 1-v)\,, \\ f_{134}(u,v) &= C_{12\bar{i}}C_{i34}\, \frac{q_1}{q}G(q_2, 2-q_4, q-1, q+1; u, 1-v)\,, \\ f_{234}(u,v) &= C_{12\bar{i}}C_{i34}\, \frac{q_2}{q}G(q_1, 2-q_3, q-1, q+1; u, 1-v)\,. \end{aligned} \quad (7.12)$$

This result is identical to that given by (5.5), (5.6), (5.7) and (5.8) for a suitable choice of the coefficient B. These terms may therefore be given solely by a chiral superfield in the operator product expansion of ϕ_1, ϕ_2 and conversely the A terms may be related to the contribution of a chiral superfield in the operator product of ϕ_3, ϕ_4.

8. Superconformal Integrals

The results obtained above are illustrated by application to integrals which automatically define superconformally covariant N-point functions. Partial results were obtained by one of us earlier [1] and a more complete analysis is undertaken here. As explained in the introduction, and as exhibited in (1.1), we may straightforwardly define a superconformally covariant scalar function for two points which is chiral at z_i and anti-chiral at z. By integration over products of such factors for $i = 1, \ldots N$, with the appropriate weighting, and integrating over z_- we may then obtain a chiral N-point function. Explicitly this gives

$$I_N(z_{1+}, \ldots z_{N+}) = \frac{i}{\pi^2} \int \mathrm{d}^4 x_- \mathrm{d}^2 \bar\theta \prod_{i=1}^{N} \frac{\Gamma(q_i)}{(x_{i+} - 2i\theta_i \sigma \bar\theta - x_-)^2} \, , \tag{8.1}$$

and the q_i are constrained by (3.2), since this is necessary to ensure that under a superconformal transformation the factors $\bar\Omega(z_-)$ from (1.1) cancel the associated transformation of the measure in (8.1). The technique for dealing with such integrals is described in appendix B and, with due account of analytic continuation to Minkowski space, the procedure described there gives, for $\Lambda = \sum_i \lambda_i$,

$$I_N(z_{1+}, \ldots z_{N+}) = \int_0^\infty \textstyle\prod_i \mathrm{d}\lambda_i \lambda_i^{q_i - 1} \displaystyle\frac{1}{\Lambda^2} \int \mathrm{d}^2\bar\theta \, \exp\Big(-\tfrac{1}{\Lambda} \textstyle\sum_{i<j} \lambda_i \lambda_j \big(x_{i+} - x_{j+} - 2i(\theta_i - \theta_j)\sigma\bar\theta \big)^2 \Big) \, . \tag{8.2}$$

The $\bar\theta$ integration is as usual performed by expanding the exponential to order $\bar\theta^2$. Since $\bar\theta$ only appears in $x_{i+} - 2i\theta_i\sigma\bar\theta$ we may write

$$I_N(z_{1+}, \ldots z_{N+}) = -\int_0^\infty \textstyle\prod_i \mathrm{d}\lambda_i \lambda_i^{q_i - 1} \displaystyle\frac{1}{\Lambda^2} \sum_{jk} \theta_j \sigma{\cdot}\partial_j \, \tilde\sigma{\cdot}\partial_k \tilde\theta_k \, e^{-\frac{1}{\Lambda}\sum_{i<j}\lambda_i\lambda_j r_{ij}} \, , \tag{8.3}$$

for r_{ij} as in (2.7). By carrying out the differentiation (8.3) becomes

$$\begin{aligned} I_N(z_{1+}, \ldots z_{N+}) = 4 \int_0^\infty & \textstyle\prod_i \mathrm{d}\lambda_i \lambda_i^{q_i - 1} \displaystyle\frac{1}{\Lambda^3} \, e^{-\frac{1}{\Lambda}\sum_{i<j}\lambda_i\lambda_j r_{ij}} \\ & \times \bigg(\sum_{ijk} \lambda_i \lambda_j \lambda_k \, \theta_i \, \mathrm{x}_{ik+} \, \tilde{\mathrm{x}}_{kj+} \, \tilde\theta_j - \tfrac{1}{2} \sum_{ij} \lambda_i \lambda_j \, r_{ij} \sum_k \lambda_k \, \theta_k^2 \bigg) \, , \end{aligned} \tag{8.4}$$

and with further rearrangement we find

$$I_N(z_{1+},\dots z_{N+}) = \, -4\int_0^\infty \textstyle\prod_i \mathrm{d}\lambda_i \lambda_i{}^{q_i-1}\,\dfrac{1}{\Lambda^3}\, e^{-\frac{1}{\Lambda}\sum_{i<j}\lambda_i\lambda_j r_{ij}} \displaystyle\sum_{i<j<k} \lambda_i\lambda_j\lambda_k r_{ij} r_{ik}\,\Lambda_{i(jk)}{}^2 \, . \qquad (8.5)$$

The integrals appearing in (8.4) and (8.5) are of the particular form considered by Symanzik [15] for conformally invariant theories. His discussion is briefly reviewed in appendix B and the essential integral, exhibited in (B.5) with (B.2), is applicable in (8.4) and (8.5) for the particular case of $\mu = 3$. For such integrals radical simplifications are possible so that for $N \geq 3$ they may be expressed in terms of functions of the harmonic cross ratios of the r_{ij}. For $N = 3$ we have from (B.6)

$$I_3(z_{1+},z_{2+},z_{3+}) = -4\,\frac{\Lambda_{1(23)}{}^2}{r_{23}}\,\frac{\Gamma(2-q_1)\Gamma(2-q_2)\Gamma(2-q_3)}{r_{23}{}^{1-q_1}\,r_{31}{}^{1-q_2}\,r_{12}{}^{1-q_3}}\,, \qquad (8.6)$$

which is in accord with the expected form in (3.3). For $N = 4$ using (B.11) and (5.12) where appropriate we have

$$\begin{aligned} I_4(z_{1+},z_{2+},z_{3+},z_{4+}) = \, -4\Bigg\{ & \frac{\Lambda_{1(23)}{}^2}{r_{23}}\left(\frac{r_{13}}{r_{23}}\right)^{q_2-1}\left(\frac{r_{12}}{r_{23}}\right)^{q_3-1}\left(\frac{r_{12}}{r_{24}}\right)^{q_4} \\ & \times H\big(q_4, 2-q_2, q_3+q_4-1, q_3+q_4+1; u,v\big) \\ & + \frac{\Lambda_{1(24)}{}^2}{r_{24}}\left(\frac{r_{24}}{r_{14}}\right)^{q_1-1}\left(\frac{r_{12}}{r_{13}}\right)^{q_3}\left(\frac{r_{12}}{r_{14}}\right)^{q_4-1} \\ & \times H\big(q_3, 2-q_1, q_3+q_4-1, q_3+q_4+1; u,v\big) \\ & + \frac{\Lambda_{1(34)}{}^2}{r_{34}}\left(\frac{r_{14}}{r_{24}}\right)^{q_2}\left(\frac{r_{14}}{r_{34}}\right)^{q_3-1}\left(\frac{r_{13}}{r_{34}}\right)^{q_4-1} \\ & \times H\big(q_2, 2-q_4, q_1+q_2-1, q_1+q_2+1; u,v\big) \\ & + \frac{\Lambda_{2(34)}{}^2}{r_{34}}\left(\frac{r_{23}}{r_{13}}\right)^{q_1}\left(\frac{r_{24}}{r_{34}}\right)^{q_3-1}\left(\frac{r_{23}}{r_{34}}\right)^{q_4-1} \\ & \times H\big(q_1, 2-q_3, q_1+q_2-1, q_1+q_2+1; u,v\big)\Bigg\}\,. \end{aligned} \qquad (8.7)$$

This result is identical, up to an overall constant, with the chiral four point function constructed in sections 3 and 4.

9. Conclusion

The results of this paper demonstrate that in some cases conformal invariance, in conjunction with supersymmetry, is sufficient to determine four point functions in four

dimensions. Nevertheless the present results are of limited direct relevance in that the unitarity bound for scalar chiral superfields $q \geq 1$ is incompatible with $\sum_i q_i = 3$ which is the case when our non trivial results are obtained. Even the three point function is applicable only if $q_i = 1$ which corresponds to free fields. However in some cases it is possible that the chiral superfields play the role of a potential for physical fields and the unitarity bound might not then apply. For $\mathcal{N} = 4$ supersymmetry there are various results which require that four and higher point functions of scalar primary superfields are just products of free fields in some extremal cases [16]. Perhaps for other cases as well superconformal symmetry may constrain the functions of the conformal cross ratios u, v while not requiring just a solution corresponding to free fields.

In this context the two variable Appell functions F_4 together with the related G and H are a natural set of functions in terms of which four point correlation functions may be expressed. The function G clearly arises in the operator product expansion and includes the contributions of all derivatives of quasi-primary fields. The properties (5.3), (5.4) and (5.10), (5.11), (5.12) are clearly what is required to ensure the essential crossing symmetry conditions on four point functions.

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Appendix A. Alternative Analysis of Four Point Functions

In terms of the discussion of section 5 we may derive an equivalent set of equations expressing superconformal invariance. We start from

$$
\begin{aligned}
&\big(x_{2+}{}^2\big)^{q_2}\big\langle\phi_1(z_{1+})\phi_2(z_{2+})\phi_3(z_{3+})\phi_4(z_{4+})\big\rangle\Big|_{\substack{x_{1+}=0,\theta_1=0\\ x_{2+}\to\infty,\theta_2=0}}\\
&= \big(x_{3+}{}^2\big)^{q_2-2}\Big(\theta_3{}^2\, f_1(v,w)+\theta_4{}^2\, f_2(v,w)+2\theta_4\tilde\theta_3\, f_3(v,w)+2\theta_4\mathrm{x}_4\tilde{\mathrm{x}}_3\tilde\theta_3\frac{1}{x_{3+}{}^2}\, f_4(v,w)\Big)\,,
\end{aligned}
\tag{A.1}
$$

for

$$
v=\frac{x_{4+}{}^2}{x_{3+}{}^2}\,,\qquad w=\frac{2x_{4+}{\cdot}x_{3+}}{x_{3+}{}^2}\,. \tag{A.2}
$$

In this particular limit we need consider only the variations in z_{3+},z_{4+} for the terms involving $\bar\epsilon,\eta$ in (2.1). From terms proportional to $i\theta_3{}^2\theta_4\mathrm{x}_4\bar\epsilon$ we have

$$
\partial_v f_1-\partial_w f_3-(v\partial_v-q_2+1)f_4=0\,, \tag{A.3}
$$

and from $i\theta_3{}^2\theta_4\mathrm{x}_3\bar\epsilon$

$$
\partial_w f_1+(v\partial_v+w\partial_w-q_2+2)f_3-v\partial_w f_4=0\,, \tag{A.4}
$$

and from $i\theta_4{}^2\theta_3\mathrm{x}_4\bar\epsilon$

$$
\partial_w f_2-\partial_v f_3-\partial_w f_4=0\,, \tag{A.5}
$$

and from $i\theta_4{}^2\theta_3\mathrm{x}_3\bar\epsilon$

$$
-(v\partial_v+w\partial_w-q_2+2)f_2-\partial_w f_3+(v\partial_v+w\partial_w+2)f_4=0\,, \tag{A.6}
$$

and from $\theta_3{}^2\theta_4\mathrm{x}_4\tilde{\mathrm{x}}_3\eta$

$$
\partial_w f_1+\partial_w f_3+(v\partial_v-q_2-q_3+2)f_4=0\,, \tag{A.7}
$$

and from $\theta_4{}^2\theta_3\mathrm{x}_3\tilde{\mathrm{x}}_4\eta$

$$
\partial_w f_2+\partial_w f_3-(v\partial_v+w\partial_w+q_4+1)f_4=0\,, \tag{A.8}
$$

and from $\theta_3{}^2\theta_4\eta$

$$
(v\partial_v+w\partial_w+q_4)f_1+(v\partial_v+w\partial_w-q_2-q_3+3)f_3-v\partial_w f_4=0\,, \tag{A.9}
$$

and from $\theta_4{}^2\theta_3\eta$

$$
-(v\partial_v-q_2-q_3+2)f_2-(v\partial_v+q_4-1)f_3-v\partial_w f_4=0\,. \tag{A.10}
$$

For the results involving η it is necessary to include the terms involving $q_3\theta_3\eta$ and $q_4\theta_4\eta$ arising from σ in the superconformal Ward identity (3.1).

The coefficient functions appearing in (A.1) may be related to those in (3.9) by

$$\begin{aligned} &f_{123} = f_1 + f_3 - v f_4 \,, \qquad v^{q_2+q_3-2} f_{124} = f_2 + f_3 - f_4 \,, \\ &u^{q_1+q_2-2} v^{q_2+q_3-2} f_{134} = f_4 \,, \qquad u^{q_1+q_2-2} f_{234} = -f_3 \,, \qquad u = 1 - w + v \,. \end{aligned} \tag{A.11}$$

With the basis in (A.1) the symmetry properties are less evident but the equations obtained here are equivalent, with (A.11), to those obtained in section 3.

Appendix B. Conformal Integrals

For completeness we recapitulate some of the results of Symanzik [15] concerning integrals defining conformally invariant N-point functions. For the purposes of this appendix we assume a Euclidean metric in general d dimensions and define

$$I_N(x_1, \dots x_N) = \frac{1}{\pi^\mu} \int \mathrm{d}^d x \prod_{i=1}^{N} \frac{\Gamma(\delta_i)}{(x - x_i)^{2\delta_i}} \,, \qquad \mu = \tfrac{1}{2} d \,, \tag{B.1}$$

and we require

$$\sum_{i=1}^{N} \delta_i = d = 2\mu \,, \tag{B.2}$$

which ensures conformal invariance. Using

$$\frac{\Gamma(\delta_i)}{(x - x_i)^{2\delta_i}} = \int_0^\infty \mathrm{d}\lambda_i \, \lambda_i^{\delta_i - 1} e^{-\lambda_i (x - x_i)^2} \,, \tag{B.3}$$

and writing $\sum_i \lambda_i (x - x_i)^2 = \Lambda \big(x - \sum_i \lambda_i x_i / \Lambda \big)^2 + \sum_{i<j} \lambda_i \lambda_j r_{ij} / \Lambda$ for

$$\Lambda = \sum_i \lambda_i \,, \qquad r_{ij} = (x_i - x_j)^2 \,, \tag{B.4}$$

we may evaluate the x integral to give

$$I_N(x_1, \dots x_N) = \int_0^\infty \textstyle\prod_i \mathrm{d}\lambda_i \, \lambda_i^{\delta_i - 1} \, \dfrac{1}{\Lambda^\mu} \, e^{-\frac{1}{\Lambda} \sum_{i<j} \lambda_i \lambda_j r_{ij}} \,. \tag{B.5}$$

The crucial observation of Symanzik is that when (B.2) holds the integral (B.5) is unchanged if we take, instead of (B.4), $\Lambda = \sum_i \kappa_i \lambda_i$ for any $\kappa_i \geq 0$, not all zero. To verify this we may make the change of variables $\lambda_i = \sigma \alpha_i$, with α_i constrained by $\sum_i \kappa_i \alpha_i = 1$.

The integration measure in (B.5), $\prod_i \mathrm{d}\lambda_i\, \lambda^{\delta_i - 1} = \prod_i \mathrm{d}\alpha_i {\alpha_i}^{\delta_i-1}\, \delta(1-\sum_i \kappa_i\alpha_i)\, \mathrm{d}\sigma\, \sigma^{d-1}$ and then, carrying out the integration over σ, the explicit dependence on Λ disappears. This result is then equivalent to making the above change in Λ.

For $N=3$ if we choose $\Lambda = \lambda_3$ in (B.5) then we may easily carry out the λ_3 integration and subsequently integrate λ_1, λ_2 to give [17]

$$I_3(x_1,x_2,x_3) = \frac{\Gamma(\mu-\delta_1)\Gamma(\mu-\delta_2)\Gamma(\mu-\delta_3)}{{r_{23}}^{\mu-\delta_1}\, {r_{31}}^{\mu-\delta_2}\, {r_{12}}^{\mu-\delta_3}}\,. \tag{B.6}$$

For $N=4$ we choose $\Lambda = \lambda_4$ and, following Symanzik [15], write

$$\begin{aligned} e^{-\frac{1}{\lambda_4}\lambda_1\lambda_2 r_{12}} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}s\ \Gamma(-s)\left(\frac{\lambda_1\lambda_2}{\lambda_4} r_{12}\right)^s, \\ e^{-\frac{1}{\lambda_4}\lambda_2\lambda_3 r_{23}} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}t\ \Gamma(-t)\left(\frac{\lambda_2\lambda_3}{\lambda_4} r_{23}\right)^t, \qquad c<0\,, \end{aligned} \tag{B.7}$$

and then we can evaluate the λ_4 and subsequently the $\lambda_1, \lambda_2, \lambda_3$ integrations to obtain

$$I_4(x_1,x_2,x_3,x_4) = {r_{14}}^{\mu-\delta_1-\delta_4}\, {r_{34}}^{\mu-\delta_3-\delta_4} {r_{13}}^{-\mu+\delta_4}\, {r_{24}}^{-\delta_2}\, \mathcal{F}_4(u,v)\,, \tag{B.8}$$

where u, v are as in (2.11) and

$$\begin{aligned} \mathcal{F}_4(u,v) = \frac{1}{(2\pi i)^2}\int & \mathrm{d}s\, \mathrm{d}t\, \Gamma(-s)\Gamma(-t)\Gamma(s+t+\delta_2)\Gamma(s+t+\mu-\delta_4) \\ & \times \Gamma(\delta_3+\delta_4-\mu-s)\Gamma(\delta_1+\delta_4-\mu-t)\, u^s v^t\,. \end{aligned} \tag{B.9}$$

By closing the contours in $Re\, s, Re\, t > 0$, using $\Gamma(x-m) = \Gamma(x)(-1)^m/(1-x)_m$, the integrals may be obtained in terms of the F_4 functions defined by (3.30)[8]

$$\begin{aligned} \mathcal{F}_4(u,v) = {} & \Gamma(\delta_2)\Gamma(\mu-\delta_4)\Gamma(\delta_3+\delta_4-\mu)\Gamma(\delta_1+\delta_4-\mu) \\ & \times F_4\big(\delta_2, \mu-\delta_4, \mu+1-\delta_3-\delta_4, \mu+1-\delta_1-\delta_4; u, v\big) \\ & + \Gamma(\mu-\delta_3)\Gamma(\delta_1)\Gamma(\delta_3+\delta_4-\mu)\Gamma(\delta_2+\delta_3-\mu) \\ & \times v^{\delta_1+\delta_4-\mu} F_4\big(\mu-\delta_3, \delta_1, \mu+1-\delta_3-\delta_4, \mu+1-\delta_2-\delta_3; u, v\big) \\ & + \Gamma(\mu-\delta_1)\Gamma(\delta_3)\Gamma(\delta_1+\delta_2-\mu)\Gamma(\delta_1+\delta_4-\mu) \\ & \times u^{\delta_3+\delta_4-\mu} F_4\big(\mu-\delta_1, \delta_3, \mu+1-\delta_1-\delta_2, \mu+1-\delta_1-\delta_4; u, v\big) \\ & + \Gamma(\delta_4)\Gamma(\mu-\delta_2)\Gamma(\delta_1+\delta_2-\mu)\Gamma(\delta_2+\delta_3-\mu) \\ & \times u^{\delta_3+\delta_4-\mu} v^{\delta_1+\delta_4-\mu} F_4\big(\delta_4, \mu-\delta_2, \mu+1-\delta_1-\delta_2, \mu+1-\delta_2-\delta_3; u, v\big)\,. \end{aligned} \tag{B.10}$$

[8] Such functions were introduced also in a related context by Ferrara *et al* in [18] and the essential integral (B.9) was considered by Davydychev and Tausk [19] who also found its expression in terms of F_4 functions.

With the aid of (5.1) and (5.9) this may be reduced just to the function H so that

$$
\begin{aligned}
I_4(x_1,x_2,x_3,x_4) &= \frac{1}{{r_{13}}^{\mu}} \left(\frac{r_{14}}{r_{24}}\right)^{\delta_2} \left(\frac{r_{14}}{r_{34}}\right)^{\delta_3-\mu} \left(\frac{r_{13}}{r_{34}}\right)^{\delta_4} \\
&\quad \times H\big(\delta_2, \mu-\delta_4, \delta_1+\delta_2+1-\mu, \delta_1+\delta_2; u, v\big) \\
&= \frac{1}{{r_{13}}^{\mu}} \left(\frac{r_{13}}{r_{23}}\right)^{\delta_2} \left(\frac{r_{12}}{r_{23}}\right)^{\delta_3-\mu} \left(\frac{r_{12}}{r_{24}}\right)^{\delta_4} \\
&\quad \times H\big(\delta_4, \mu-\delta_2, \delta_3+\delta_4+1-\mu, \delta_3+\delta_4; u, v\big) \, ,
\end{aligned}
\tag{B.11}
$$

where the two expressions are related by (5.12).[9]

Appendix C. Particular Results for G and H

The two variable functions F_4, and also G, H given by (5.1), (5.9), are relatively unfamiliar although H occurs in various contexts in the evaluation of Feynman integrals [19,20]. In this appendix, we list some results for G which are relevant and give formulae for special cases, when they reduce to well known single variable functions.

From the definition (5.1) we may directly obtain

$$
\begin{aligned}
\partial_u G(\alpha,\beta,\gamma,\delta;u,1-v) &= \frac{\alpha\beta(\delta-\alpha)(\delta-\beta)}{\gamma\delta(\delta+1)} \\
&\qquad \times G(\alpha+1,\beta+1,\gamma+1,\delta+2;u,1-v) \, , \\
\partial_v G(\alpha,\beta,\gamma,\delta;u,1-v) &= -\,\frac{\alpha\beta}{\delta}\, G(\alpha+1,\beta+1,\gamma,\delta+1;u,1-v) \, , \\
\big(\beta + u\partial_u + v\partial_v\big) G(\alpha,\beta,\gamma,\delta;u,1-v) &= \frac{\beta(\delta-\alpha)}{\delta}\, G(\alpha,\beta+1,\gamma,\delta+1;u,1-v) \, .
\end{aligned}
\tag{C.1}
$$

There are also various recurrence relations:

$$
\begin{aligned}
\delta(\delta-\gamma-\beta+1)G(\alpha,\beta,\gamma,\delta;u,1-v) = &\; -\delta(\gamma-1)G(\alpha,\beta,\gamma-1,\delta;u,1-v) \\
&\; +\alpha(\delta-\beta)G(\alpha+1,\beta,\gamma,\delta+1;u,1-v) \\
&\; +(\delta-\alpha)(\delta-\beta)G(\alpha,\beta,\gamma,\delta+1;u,1-v) \, ,
\end{aligned}
\tag{C.2a}
$$

$$
\begin{aligned}
\delta G(\alpha,\beta,\gamma,\delta;u,1-v) = \beta G(\alpha,\beta+1,\gamma,\delta+1;u,1-v) + (\delta-\beta)G(\alpha,\beta,\gamma,\delta+1;u,1-v) \\
-\frac{\alpha\beta(\delta-\beta)}{\gamma(\delta+1)}\, uG(\alpha+1,\beta+1,\gamma+1,\delta+2;u,1-v) \, ,
\end{aligned}
\tag{C.2b}
$$

[9] By considering the limit ${x_4}^2 \to \infty$, when $u = r_{12}/r_{13}$, $v = r_{23}/r_{13}$, this also gives an expression for the integral I_3 in (B.1) without the condition (B.2) which then determines δ_4 in (B.11).

$$
\begin{aligned}
(\beta-\alpha)\delta G(\alpha,\beta,\gamma,\delta;u,1-v) &= (\delta-\alpha)\beta G(\alpha,\beta+1,\gamma,\delta+1;u,1-v) \\
&\quad - (\delta-\beta)\alpha G(\alpha+1,\beta,\gamma,\delta+1;u,1-v)\,, \qquad (\text{C.2}c) \\
\delta(\delta-\alpha-\beta)G(\alpha,\beta,\gamma,\delta;u,1-v) &= (\delta-\alpha)(\delta-\beta)G(\alpha,\beta,\gamma,\delta+1;u,1-v) \\
&\quad - \alpha\beta\, vG(\alpha+1,\beta+1,\gamma,\delta+1;u,1-v)\,. \qquad (\text{C.2}d)
\end{aligned}
$$

For the particular case when $\delta=\gamma+n$, $n=0,1,\dots$ the G-function may be reduced to products of ordinary hypergeometric functions. Defining[10]

$$
u = x(1-y)\,, \qquad v = y(1-x)\,, \tag{C.3}
$$

these originate from the reduction formula for F_4

$$
F_4(\alpha,\beta,\gamma,\gamma';u,v) = F(\alpha,\beta;\gamma;x)F(\alpha,\beta;\gamma';y)\,, \quad \alpha+\beta=\gamma+\gamma'-1\,. \tag{C.4}
$$

Applying this in (5.2) and using standard hypergeometric identities gives

$$
G(\alpha,\beta,\gamma,\gamma;u,1-v) = F(\alpha,\beta;\gamma;x)F(\alpha,\beta;\gamma;1-y)\,. \tag{C.5}
$$

Using (C.1) for $\partial_v G$ then gives

$$
\begin{aligned}
&G(\alpha,\beta,\gamma,\gamma+1;u,1-v) \\
&\quad = \frac{1}{1-x-y}\big((1-y)F(\alpha-1,\beta-1;\gamma;x)F(\alpha,\beta;\gamma+1;1-y) \\
&\qquad\qquad - xF(\alpha,\beta;\gamma+1;x)F(\alpha-1,\beta-1;\gamma;1-y)\big) \\
&\quad = \frac{1}{1-x-y}\frac{\gamma}{\alpha-1}\big(F(\alpha-1,\beta-1;\gamma;x)F(\alpha-1,\beta;\gamma;1-y) \\
&\qquad\qquad - F(\alpha,\beta-1;\gamma;x)F(\alpha-1,\beta-1;\gamma;1-y)\big) \\
&\quad = \frac{1}{1-x-y}\frac{\gamma(\gamma-1)}{(\alpha-1)(\beta-1)}\big(F(\alpha-1,\beta-1;\gamma;x)F(\alpha-1,\beta-1;\gamma-1;1-y) \\
&\qquad\qquad - F(\alpha-1,\beta-1;\gamma-1;x)F(\alpha-1,\beta-1;\gamma;1-y)\big)\,.
\end{aligned} \tag{C.6}
$$

The results for the four point functions obtained here are given by the G-function for

[10] To invert we may define $\lambda=((1-u-v)^2-4uv)^{\frac12}=1-x-y$ and $\rho=2/(1-u-v+\lambda)=1/((1-x)(1-y))$ and then $x=\rho u/(1+\rho u)$, $y=\rho v/(1+\rho v)$.

$\delta = \gamma + 2$. Following a similar route as which led to (C.6) we may obtain

$$
\begin{aligned}
&G(\alpha+1,\beta+1,\gamma,\gamma+2;u,1-v) \\
&\quad = \frac{1}{(1-x-y)^2}\Big((1-y)^2 F(\alpha-1,\beta-1;\gamma;x)F(\alpha+1,\beta+1;\gamma+2;1-y) \\
&\qquad\qquad\qquad + x^2 F(\alpha+1,\beta+1;\gamma+2;x)F(\alpha-1,\beta-1;\gamma;1-y)\Big) \\
&\quad - 2\frac{\gamma+1}{\alpha\beta}\,\frac{x(1-y)}{(1-x-y)^3} \\
&\quad \times \Big(\big(\gamma F(\alpha{-}1,\beta{-}1;\gamma;x) - (\gamma-1)F(\alpha{-}1,\beta{-}1;\gamma{-}1;x)\big)F(\alpha,\beta;\gamma{+}1;1-y) \\
&\qquad - F(\alpha,\beta;\gamma{+}1;x)\big(\gamma F(\alpha{-}1,\beta{-}1;\gamma;1-y) - (\gamma-1)F(\alpha{-}1,\beta{-}1;\gamma{-}1;1-y)\big)\Big)\,.
\end{aligned} \tag{C.7}
$$

The result (C.5) is relevant, according to (B.10), in two dimensions. Using complex coordinates for this case and defining

$$
\eta = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}\,, \tag{C.8}
$$

we have

$$
u = \eta\bar\eta\,, \qquad v = (1-\eta)(1-\bar\eta)\,, \tag{C.9}
$$

and then from (C.5) we have

$$
G(\alpha,\beta,\gamma,\gamma;u,1-v) = F(\alpha,\beta;\gamma;\eta)F(\alpha,\beta;\gamma;\bar\eta)\,, \tag{C.10}
$$

exhibiting a holomorphic factorisation. For the crossing symmetric function H given by (5.9) we have

$$
\begin{aligned}
H(\alpha,\beta,\gamma,\gamma;u,v) &= \frac{\pi}{\sin\pi\gamma}\Big\{N_1\big|F(\alpha,\beta;\gamma;\eta)\big|^2 \\
&\qquad\qquad + N_2\big|\eta^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\eta)\big|^2\Big\}\,, \\
N_1 = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma)^2}&\,, \quad N_2 = -\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)}{\Gamma(2-\gamma)^2}\,,
\end{aligned} \tag{C.11}
$$

which coincides with expressions previously obtained for two dimensional conformal four point functions [21].

We may also apply (C.6) to cases of relevance in four dimensions. It is easy to see that it gives

$$
G(1,1,1,2;u,1-v) = -\frac{1}{1-x-y}\ln\frac{y}{1-x}\,. \tag{C.12}
$$

The definition (5.9) for H gives

$$
\begin{aligned}
H(1,1,1+\epsilon,2+\epsilon;u,v) = \frac{\pi}{\sin\pi\epsilon}\bigg(& -\frac{1}{1+\epsilon} G(1,1,1+\epsilon,2+\epsilon;u,1-v) \\
& + \frac{1}{1-\epsilon}\Big(\frac{v}{u}\Big)^{\epsilon} G(1,1,1-\epsilon,2-\epsilon;u,1-v)\bigg)\,,
\end{aligned}
\tag{C.13}
$$

and using (C.6) with

$$
wF(1,1;2+\epsilon;w) = -(1+\epsilon)\ln(1-w) - \epsilon\Big(\tfrac{1}{2}\big(\ln(1-w)\big)^2 + \mathrm{Li}_2(w)\Big) + \mathrm{O}(\epsilon^2)\,, \tag{C.14}
$$

where Li_2 is the dilogarithm function, we have

$$
H(1,1,1,2;u,v) = \frac{1}{1-x-y}\Big(\ln x(1-y)\,\ln\frac{y}{1-x} - 2\mathrm{Li}_2(x) + 2\mathrm{Li}_2(1-y)\Big)\,. \tag{C.15}
$$

This result is relevant acording to (B.11) if $N, d = 4$ in (B.1) and we assume $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$ and is equivalent with the the results given in [19,20] for this integral.

It is of interest to verify that (C.15) satisfies the symmetry relations (5.10) and (5.11). For $u \leftrightarrow v$ when $x \leftrightarrow y$ we may use $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{1}{6}\pi^2 - \ln x\,\ln(1-x)$ whereas when $u \to u' = u/v$, $v \to v' = 1/v$ we have $x' = 1 - 1/y$, $y' = 1/(1-x)$ we may use the identity $\mathrm{Li}_2(x) + \mathrm{Li}_2(x/(x-1)) = -\frac{1}{2}(\ln(1-x))^2$.

Appendix D. Operator Product Expansion Calculations

We describe here some details of the calculations involved in applying the operator product expansion to the four point function. The essential derivative operator defined by (7.4) is given explicitly by

$$
\begin{aligned}
C_\kappa^{a,b}(s,\partial) &= \frac{1}{B(a,b)}\sum_{n=0}\frac{1}{n!(\kappa)_n}\big(-\tfrac{1}{4}s^2\partial^2\big)^n\int_0^1 \mathrm{d}\alpha\,\alpha^{a+n-1}(1-\alpha)^{b+n-1}e^{\alpha s\cdot\partial} \\
&= \frac{1}{B(a,b)}\sum_{m,n=0}\frac{(a)_{m+n}(b)_n}{m!n!(a+b)_{m+2n}(\kappa)_n}\,\big(s\cdot\partial\big)^m\big(-\tfrac{1}{4}s^2\partial^2\big)^n\,.
\end{aligned}
\tag{D.1}
$$

To verify (7.4) we may use

$$
\big(\tfrac{1}{4}\partial_{x_2}{}^2\big)^n\frac{1}{r_{23}{}^S} = (S)_n(S+1-\tfrac{1}{2}d)_n\frac{1}{r_{23}{}^{S+n}}\,, \tag{D.2}
$$

so that

$$
C_{S+1-\frac{1}{2}d}^{a,b}(x_{12},\partial_{x_2})\frac{1}{r_{23}{}^S} = \frac{1}{B(a,b)}\int_0^1 \mathrm{d}\alpha\,\alpha^{a-1}(1-\alpha)^{b-1}\sum_{n=0}\frac{(S)_n}{n!}\frac{\big(-\alpha(1-\alpha)r_{12}\big)^n}{\big(x_{23}+\alpha x_{12}\big)^{2(S+n)}}\,. \tag{D.3}
$$

With $(x_{23}+\alpha x_{12})^2=(1-\alpha)r_{23}+\alpha r_{13}-\alpha(1-\alpha)r_{12}$ the sum over n is straightforward, giving $\big((1-\alpha)r_{23}+\alpha r_{13}\big)^{-S}$, and then the α integration, with $S=a+b$, is of the form

$$\int_0^1 \mathrm{d}\alpha\, \alpha^{a-1}(1-\alpha)^{b-1}\frac{1}{\big(\alpha s+(1-\alpha)t\big)^{a+b}}=B(a,b)\,\frac{1}{s^a t^b}\,, \tag{D.4}$$

giving the desired result.

To sketch the calculation leading to (7.5) we follow the method described in [22] (although related results were obtained long ago in [23]) and first write, by applying similar methods as which led to (D.3),

$$\begin{aligned} &C^{a,b}_{S+1-\frac{1}{2}d}(x_{12},\partial_{x_2})\frac{1}{{r_{23}}^{e}{r_{24}}^{f}} \\ &\qquad = \frac{1}{B(a,b)B(e,f)}\int_0^1 \mathrm{d}\alpha\,\alpha^{a-1}(1-\alpha)^{b-1}\int_0^1 \mathrm{d}\beta\,\beta^{e-1}(1-\beta)^{f-1} \\ &\qquad\quad \times\sum_{m,n=0}\frac{(S)_{m+n}(S+1-\frac{1}{2}d)_{m+n}}{m!n!(S+1-\frac{1}{2}d)_m(S+1-\frac{1}{2}d)_n}\frac{A^m B^n}{\big(\alpha x_{12}-\beta x_{34}+x_{24}\big)^{2(S+m+n)}}\,, \\ &\qquad\qquad A=-\alpha(1-\alpha)r_{12}\,, \qquad\qquad B=-\beta(1-\beta)r_{34}\,. \end{aligned} \tag{D.5}$$

By using $(\alpha x_{12}-\beta x_{34}+x_{24})^2=A+B+C$ where $C=\alpha(1-\beta)r_{14}+\beta(1-\alpha)r_{23}+\alpha\beta r_{13}+(1-\alpha)(1-\beta)r_{24}$ we may express the second line on the r.h.s. of (D.5) in the form

$$\frac{1}{C^S}\sum_{m=0}\frac{(S)_{2m}}{m!(S+1-\frac{1}{2}d)_m}\Big(\frac{AB}{C^2}\Big)^m\,. \tag{D.6}$$

The α integration is just as in (D.4) and the β integration may be carried out using

$$\int_0^1 \mathrm{d}\beta\,\beta^{e-1}(1-\beta)^{f-1}\frac{1}{(1-\beta x)^a(1-\beta y)^b}=B(e,f)\frac{1}{(1-x)^e}\sum_{n=0}\frac{(e)_n(b)_n}{n!(S)_n}\Big(\frac{y-x}{1-x}\Big)^n\,, \tag{D.7}$$

where $S=a+b=e+f$ and we apply this result for $x=1-r_{13}/r_{14}$, $y=1-r_{23}/r_{24}$ so that $(y-x)/(1-x)=1-v$. Combining these expressions gives (7.5) with G given by the series in (5.1).

These results simplify in two dimensions since, using complex coordinates, (D.1) factorises,

$$C^{a,b}_S(s,\partial)={}_1F_1(a;S;s_z\partial_z)\,{}_1F_1(a;S;s_{\bar z}\partial_{\bar z})\,,\qquad S=a+b\,, \tag{D.8}$$

since

$${}_1F_1(a;S;z_{12}\partial_{z_2})\frac{1}{{z_{23}}^{S}}=\frac{1}{{z_{12}}^{a}\,{z_{23}}^{b}}\,. \tag{D.9}$$

For this case

$$
{}_1F_1(a;S;z_{12}\partial_{z_2})\frac{1}{{z_{23}}^e\,{z_{24}}^f}=\frac{1}{{z_{14}}^a\,{z_{24}}^b}\Big(\frac{z_{14}}{z_{13}}\Big)^e F(b,e;S;\eta)\,, \tag{D.10}
$$

with η defined by (C.8). This result is in accord with the holomorphic factorisation required by (C.5).

For application to the supersymmetric case we also consider, when $d=4$, the extension to spinor fields. For this we require

$$
C^{a,b}(x_{12},\partial_{x_2})_\alpha{}^\beta\,\frac{(\mathrm{x}_{23})_{\beta\dot\alpha}}{{r_{23}}^{S+1}}=\frac{(\mathrm{x}_{13})_{\alpha\dot\alpha}}{{r_{13}}^{a+1}{r_{23}}^{b}}\,. \tag{D.11}
$$

This has the solution

$$
C^{a,b}(x_{12},\partial_{x_2})_\alpha{}^\beta=C^{a+1,b}_{S-1}(x_{12},\partial_{x_2})\delta_\alpha{}^\beta+\frac{b}{S^2-1}C^{a+1,b+1}_{S}(x_{12},\partial_{x_2})\tfrac{1}{2}(\mathrm{x}_{12}\tilde\partial_{x_2})_\alpha{}^\beta\,. \tag{D.12}
$$

and, in a similar fashion to the derivation of (7.5),

$$
\begin{aligned}
&C^{a,b}(x_{12},\partial_{x_2})_\alpha{}^\beta\,\frac{(\mathrm{x}_{23})_{\beta\dot\alpha}}{{r_{23}}^{e+1}{r_{24}}^{f}}\\
&\quad=\frac{1}{{r_{14}}^{a+1}{r_{24}}^{b}}\Big(\frac{r_{14}}{r_{13}}\Big)^{e+1}\Big\{(\mathrm{x}_{13})_{\alpha\dot\alpha}G(b,e+1,S-1,S+1;u,1-v)\\
&\qquad+\frac{bf}{S^2-1}\,(\mathrm{x}_{12}{\mathrm{x}_{42}}^{-1}\mathrm{x}_{43})_{\alpha\dot\alpha}G(b+1,e+1,S,S+2;u,1-v)\Big\}\,.
\end{aligned} \tag{D.13}
$$

Appendix E. Supersymmetric Calculations

The application of the operator product expansion to the supersymmetric case requires an extension of the previous results. We first derive the expression (7.10) which is required to satisfy (7.9). To achieve this we use

$$
\frac{1}{\big(x-2i\theta_{12}\sigma\bar\theta\big)^{2q_1}}=\frac{1}{x^{2q_1}}+4iq_1\,\frac{\theta_{12}\mathrm{x}\bar\theta}{x^{2(q_1+1)}}+4q_1(q_1-1)\,\frac{{\theta_{12}}^2\bar\theta^2}{x^{2(q_1+1)}}\,, \tag{E.1}
$$

in (7.9) with $x=x_{1+}-2i\theta_2\sigma\bar\theta-x_-$. The results (7.4) and (D.11) then show, since $-\frac{1}{4}{\partial_{\theta_2}}^2{\theta_2}^2=1$, that (7.9) is satisfied if

$$
\begin{aligned}
\mathcal{C}^{q_1,q_2}(z_{12+},\partial_{z_{2+}})=&\;C^{q_1,q_2}_{q-1}(x_{12+},\partial_{x_{2+}})+\frac{q_1}{q}\,\theta_{12}C^{q_1,q_2}(x_{12+},\partial_{x_{2+}})\partial_{\theta_2}\\
&-\frac{q_1(q_1-1)}{4q(q-1)}{\theta_{12}}^2C^{q_1+1,q_2}_{q}(x_{12+},\partial_{x_{2+}}){\partial_{\theta_2}}^2\,,
\end{aligned} \tag{E.2}
$$

which is identical with (7.10).

The results in (7.12) are obtained by calculating

$$\mathcal{C}^{q_1,q_2}(z_{12+},\partial_{z_{2+}})\frac{{\Lambda_{2(34)}}^2}{{r_{23}}^{1-q_4}{r_{24}}^{1-q_3}}\,{r_{34}}^{q-2}\,. \tag{E.3}$$

The terms $\mathrm{O}({\theta_{12}}^2)$ are easily seen, using (7.5), to be

$$\begin{aligned}&\frac{q_1(q_1-1)}{q(q-1)}\,{\theta_{12}}^2\,C_q^{q_1+1,q_2}(x_{12+},\partial_{x_{2+}})\frac{{r_{34}}^{q-1}}{{r_{23}}^{1-q_4}{r_{24}}^{1-q_3}}\\&=\frac{q_1(q_1-1)}{q(q-1)}\,{\theta_{12}}^2\frac{{r_{34}}^{q-1}}{{r_{14}}^{q_1+1}{r_{24}}^{q_2}}\Big(\frac{r_{14}}{r_{13}}\Big)^{2-q_4}G(2-q_4,q_2,q,q+1;u,1-v)\,,\end{aligned} \tag{E.4}$$

where, using (5.3) and (C.2a),

$$\begin{aligned}\frac{1-q_1}{q_2}G(2-q_4,q_2,q,q+1;u,1-v)&=\frac{1-q_1}{q_2}v^{q_1+q_4-1}G(2-q_3,q_1+1,q,q+1;u,1-v)\\&=\frac{2-q_4}{q+1}\,v^{q_1+q_4-1}G(2-q_3,q_1+1,q,q+2;u,1-v)\\&\quad+\frac{2-q_3}{q+1}\,G(2-q_4,q_2+1,q,q+2;u,1-v)\\&\quad+\frac{1}{q_2}(1-q)\,G(2-q_4,q_2,q-1,q+1;u,1-v)\,.\end{aligned} \tag{E.5}$$

The $\mathrm{O}(\theta_{12})$ terms arise from

$$2\frac{q_1}{q}\,\theta_{12}C^{q_1,q_2}(x_{12+},\partial_{x_{2+}})\frac{\tilde{\theta}_{23}+\mathrm{x}_{23+}\tilde{\ell}_{34}}{{r_{23}}^{2-q_4}{r_{24}}^{2-q_3}}\,{r_{34}}^{q-1}\,, \tag{E.6}$$

with $C^{q_1,q_2}(x_{12+},\partial_{x_{2+}})$ given by (D.12). For the terms not involving θ_{23} (D.13) gives

$$\begin{aligned}2\frac{q_1}{q}\,\frac{{r_{34}}^{q-1}}{{r_{14}}^{q_1+1}{r_{24}}^{q_2}}\Big(\frac{r_{14}}{r_{13}}\Big)^{2-q_4}\Big\{&\theta_{12}\mathrm{x}_{13+}\tilde{\ell}_{34}\,G(q_2,2-q_4,q-1,q+1;u,1-v)\\&+\frac{q_2(2-q_3)}{q^2-1}\,\theta_{12}\mathrm{x}_{12+}{\mathrm{x}_{42+}}^{-1}\mathrm{x}_{43+}\tilde{\ell}_{34}\,G(q_2+1,2-q_4,q,q+2;u,1-v)\Big\}\,.\end{aligned} \tag{E.7}$$

The remaining terms involving θ_{23} may be calculated from this using

$$\frac{(\mathrm{x}_{23+})_{\beta\dot{\beta}}}{{r_{23}}^{2-q_4}}\,(\tilde{\sigma}\cdot\overleftarrow{\partial}_{x_{3+}})^{\dot{\beta}\alpha}=2q_4\,\frac{1}{{r_{23}}^{2-q_4}}\,{\delta_\beta}^\alpha\,. \tag{E.8}$$

Using both the derivative relations (C.1) and (C.2b) we get

$$
\begin{aligned}
2\frac{q_1}{q}\,\frac{{r_{34}}^{q-1}}{{r_{14}}^{q_1+1}{r_{24}}^{q_2}}\left(\frac{r_{14}}{r_{13}}\right)^{2-q_4}&\Big\{\theta_{12}\tilde\theta_{23}\,G(q_2,2-q_4,q-1,q+1;u,1-v)\\
&+\frac{q_2(2-q_4)}{q^2-1}\,\theta_{12}\mathrm{x}_{12+}{\mathrm{x}_{32+}}^{-1}\tilde\theta_{23}\,v^{q_1+q_4-1}G(q_1+1,2-q_3,q,q+2;u,1-v)\\
&+\frac{q_2(2-q_3)}{q^2-1}\,\theta_{12}\mathrm{x}_{12+}{\mathrm{x}_{42+}}^{-1}\tilde\theta_{23}\,G(q_2+1,2-q_4,q,q+2;u,1-v)\Big\}\,.
\end{aligned}
\tag{E.9}
$$

The remaining terms arise from

$$
C^{q_1,q_2}_{q-1}(x_{12+},\partial_{x_{2+}})\frac{{\Lambda_{2(34)}}^2}{{r_{23}}^{1-q_4}{r_{24}}^{1-q_3}}\,{r_{34}}^{q-2}\,,
\tag{E.10}
$$

where it is convenient to write

$$
{\Lambda_{2(34)}}^2=\frac{1}{r_{23}r_{24}}\big({\theta_{23}}^2r_{24}+({\theta_{24}}^2-2\theta_{23}\tilde\theta_{24})r_{23}+2\theta_{23}\mathrm{x}_{23+}\tilde{\mathrm{x}}_{34+}\tilde\theta_{24}\big)\,.
\tag{E.11}
$$

The ${\theta_{23}}^2$ and ${\theta_{24}}^2$ terms are found from (7.5) to be

$$
\frac{{r_{34}}^{q-2}}{{r_{14}}^{q_1}{r_{24}}^{q_2}}\left(\frac{r_{14}}{r_{13}}\right)^{1-q_4}\Big\{\frac{r_{14}}{r_{13}}{\theta_{23}}^2G(2-q_4,q_2,q-1,q;u,1-v)+{\theta_{24}}^2G(1-q_4,q_2,q-1,q;u,1-v)\Big\}\,.
\tag{E.12}
$$

Using (5.3) and (C.2b) we may rewrite the G-functions as

$$
\begin{aligned}
G(2-q_4,q_2,&q-1,q;u,1-v)=\frac{q_1}{q}G(2-q_4,q_2,q-1,q+1;u,1-v)\\
&+\frac{q_2}{q}v^{q_1+q_4-2}G(2-q_3,q_1,q-1,q+1;u,1-v)\\
&-\frac{q_1q_2(2-q_4)}{q(q^2-1)}\,uv^{q_1+q_4-2}G(2-q_3,q_1+1,q,q+2;u,1-v)\,,\\
G(1-q_4,q_2,&q-1,q;u,1-v)=\frac{q_1}{q}G(2-q_4,q_2,q-1,q+1;u,1-v)\\
&+\frac{q_2}{q}v^{q_1+q_4-1}G(2-q_3,q_1,q-1,q+1;u,1-v)\\
&-\frac{q_1q_2(2-q_3)}{q(q^2-1)}\,uG(2-q_4,q_2+1,q,q+2;u,1-v)\,.
\end{aligned}
\tag{E.13}
$$

For the other terms present in (E.11) we may use

$$
\frac{\mathrm{x}_{23}}{{r_{23}}^{2-q_4}}=\frac{1}{2(1-q_4)}\,\frac{1}{{r_{23}}^{1-q_4}}\sigma\cdot\overleftarrow{\partial}_3\,.
\tag{E.14}
$$

From the derivative of the terms $\propto {\theta_{24}}^2$ in (E.12) we then obtain

$$2\frac{{r_{34}}^{q-2}}{{r_{14}}^{q_1}{r_{24}}^{q_2}}\left(\frac{r_{14}}{r_{13}}\right)^{1-q_4}\bigg\{\theta_{23}\mathrm{x}_{13+}{\mathrm{x}_{41+}}^{-1}\tilde{\theta}_{24}\,\frac{q_1}{q}G(2-q_4,q_2,q-1,q+1;u,1-v) \\ +\theta_{23}\mathrm{x}_{23+}{\mathrm{x}_{42+}}^{-1}\tilde{\theta}_{24}\,\frac{q_2}{q}G(2-q_4,q_2+1,q-1,q+1;u,1-v)\bigg\}\,. \tag{E.15}$$

The results (E.4), with (E.5), (E.7), (E.9), (E.12), with (E.13), and (E.15) give exactly the superconformal form with (7.12).

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